1.7 Demuestre que la fórmula general para la sucesiuón de Fibonacci es:

$$(\forall n \in \mathbb{N}) \left(a_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}} \right).$$

Solución: Sabemos que la sucesión de Fibonacci está dada en forma recurrente por:

$$\begin{aligned} F_1 &= 1, \\ F_2 &= 1, \\ F_{n+2} &= F_{n+1} + F_n, \text{ para } n \in \mathbb{N}. \end{aligned}$$

Es fácil de verificar que $a_1 = a_2 = 1$, para concluir el ejercicio veamos que:

$$(\forall n \in \mathbb{N}) \left(a_{n+2} = a_{n+1} + a_n \right),$$

o equivalentemente que

$$(\forall n \in \mathbb{N}) \left(a_n = a_{n+2} - a_{n+1} \right).$$

Bien, sea $n \in \mathbb{N}$ arbitrario, entonces

$$\begin{aligned} a_{n+2} - a_{n+1} &= \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}} - \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}} \\ &= \frac{\left[\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}\right] - 2\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right]}{2^{n+2}\sqrt{5}} \\ &= \frac{\left(1+\sqrt{5}\right)^{n+1}\left[\left(1+\sqrt{5}\right) - 2\right] + \left(1-\sqrt{5}\right)^{n+1}\left[\left(\sqrt{5}-1\right) + 2\right]}{2^{n+2}\sqrt{5}} \\ &= \frac{\left(1+\sqrt{5}\right)^{n+1}\left[\sqrt{5}-1\right] + \left(1-\sqrt{5}\right)^{n+1}\left[\sqrt{5}+1\right]}{2^{n+2}\sqrt{5}} \\ &= \frac{\left(1+\sqrt{5}\right)^{n}\left(1+\sqrt{5}\right)\left(\sqrt{5}-1\right) + \left(1-\sqrt{5}\right)^{n}\left(1-\sqrt{5}\right)\left(\sqrt{5}+1\right)}{2^{n+2}\sqrt{5}} \\ &= \frac{\left(1+\sqrt{5}\right)^{n}(5-1) + \left(1-\sqrt{5}\right)^{n}(1-5)}{2^{n+2}\sqrt{5}} \\ &= \frac{4\left(1+\sqrt{5}\right)^{n} - 4\left(1-\sqrt{5}\right)^{n}}{2^{n+2}\sqrt{5}} = \frac{\left(1+\sqrt{5}\right)^{n} - \left(1-\sqrt{5}\right)^{n}}{2^{n}\sqrt{5}} = \\ &= a_n, \end{aligned}$$

como se queria demostrar.