$f(x) = 0$? Show that there is a largest x in $[a, b]$ with $f(x) = 0$. (Try to give an easy proof by considering a new function closely related to f.)

(b) The proof of Theorem 1 depended upon consideration of $A = \{x : a \leq x \leq b$ and f is negative on $[a, x]\}$. Give another proof of Theorem 1, which depends upon consideration of $B = \{x : a \leq x \leq b$ and $f(x) < 0\}$. Which point x in $[a, b]$ with $f(x) = 0$ will this proof locate? Give an example where the sets A and B are not the same.

***4.** (a) Suppose that f is continuous on $[a, b]$ and that $f(a) = f(b) = 0$. Suppose also that $f(x_0) > 0$ for some x_0 in $[a, b]$. Prove that there are numbers c and d with $a \leq c < x_0 < d \leq b$ such that $f(c) = f(d) = 0$, but $f(x) > 0$ for all x in (c, d). Hint: The previous problem can be used to good advantage.

(b) Suppose that f is continuous on $[a, b]$ and that $f(a) < f(b)$. Prove that there are numbers c and d with $a \leq c < d \leq b$ such that $f(c) = f(a)$ and $f(d) = f(b)$ and $f(a) < f(x) < f(d)$ for all x in (c, d).

5. (a) Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. Hint: Let l by the largest integer satisfying $l \leq x$, and consider $l + 1$.

(b) Suppose $x < y$. Prove that there is a rational number r such that $x < r < y$. Hint: If $1/n < y - x$, then $ny - nx > 1$. (Query: Why have parts (a) and (b) been postponed until this problem set?)

(c) Suppose that $r < s$ are rational numbers. Prove that there is an irrational number between r and s. Hint: As a start, you know that there is an irrational number between 0 and 1.

(d) Suppose that $x < y$. Prove that there is an irrational number between x and y. Hint: It is unnecessary to do any more work; this follows from (b) and (c).

***6.** A set A of real numbers is said to be **dense** if every open interval contains a point of A. For example, Problem 5 shows that the set of rational numbers and the set of irrational numbers are each dense.

(a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A, then $f(x) = 0$ for all x.

(b) Prove that if f and g are continuous and $f(x) = g(x)$ for all x in a dense set A, then $f(x) = g(x)$ for all x.

(c) If we assume instead that $f(x) \geq g(x)$ for all x in A, show that $f(x) \geq g(x)$ for all x. Can $\geq$ be replaced by $>$ throughout?

7. Prove that if f is continuous and $f(x + y) = f(x) + f(y)$ for all x and y, then there is a number c such that $f(x) = cx$ for all x. (This conclusion can be demonstrated simply by combining the results of two previous problems.) Point of information: There *do* exist *noncontinuous* functions f satisfying $f(x + y) = f(x) + f(y)$ for all x and y, but we cannot prove this now; in fact, this simple question involves ideas that are usually never mentioned in any undergraduate course. The Suggested Reading contains references.