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Distributivity of Quotients of Countable Products of Boolean Algebras

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Abstract. *We compute the distributivity numbers of algebras of the type* $\mathbb{B}^\omega / \mathrm{Fin}$ *where* $\mathbb{B}$ *is the trivial algebra* $\{0,1\}$*, the countable atomless Boolean algebra,* $\mathcal{P}(\omega)$*,* $\mathcal{P}(\omega)/\mathrm{fin}$ *and* $(\mathcal{P}(\omega)/\mathrm{fin})^\omega$*.*

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1. Introduction

Given a Boolean algebra $\mathbb{B}$, the completion of $\mathbb{B}$ is denoted by r.o. $(\mathbb{B})$. Formally, r.o. $(\mathbb{B})$ is defined as the Boolean algebra of regular open subsets of $\mathbb{B}$ (see [12, p. 152]). Given a cardinal κ, r.o. $(\mathbb{B})$ is called κ-distributive if and only if the equality

$$\prod\left\{\sum_{i\in I_\alpha} u_{\alpha,i} : \alpha<\kappa\right\} = \sum\left\{\prod u_{\alpha,f(\alpha)} : f\in\prod_{\alpha<\kappa} I_\alpha\right\}$$

holds for every family $\langle u_{\alpha,i} : i\in I_\alpha \ \&\ \alpha<\kappa\rangle$ of members of $\mathbb{B}$. It is well known (see [12, p. 158]) that the following four statements are equivalent:

1. $\mathbb{B}$ is κ-distributive.
2. The intersection of κ open dense sets in $\mathbb{B}^+$ $(=\mathbb{B}\setminus\{0\})$ is dense.
3. Every family of κ maximal antichains of $\mathbb{B}^+$ has a refinement.
4. Forcing with $\mathbb{B}$ does not add a new subset of κ.

The *distributivity number* of $\mathbb{B}$ is defined as the least κ such that r.o. $(\mathbb{B})$ is not κ-distributive. The distributivity number of $\mathbb{B}$ is usually denoted by $\mathfrak{h}(\mathbb{B})$.

We are interested in computing the distributivity number of algebras of the type $\mathbb{B}^\omega/\mathrm{Fin}$. Here, $\mathbb{B}^\omega$ is the Boolean algebra of all functions $f:\omega\to\mathbb{B}$ with pointwise operation. As usual, the *support* of an element $f \in \mathbb{B}^{\omega}$ is the set of all $n \in \omega$ for which $f(n) \neq 0 \in \mathbb{B}$. Finally, Fin is the ideal of all functions with finite support and $\mathbb{B}^{\omega}/\operatorname{Fin}$ is the quotient algebra.

Boolean algebras of the type $\mathbb{B}^{\omega}/\operatorname{Fin}$ have been recently an object of study, see for example [9], [1], [5], [4]. We are going to focus in some of the most natural algebras $\mathbb{B}$ such as $\{0,1\}$, $\mathcal{P}(\omega)$, $\mathcal{P}(\omega)/\mathrm{fin}$ and the atomless countable Boolean algebra. The algebra $\mathbb{B}^{\omega}/\operatorname{Fin}$ for these Boolean algebras correspond to the Stone-Čech remainders ($X^{*} = \beta X \setminus X$) of some well known spaces. It is easy to see that $\{0,1\}^{\omega}/\mathrm{Fin}$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$ and it is well known that its distributivity number is denoted by $\mathfrak{h}$, that $\aleph_1 \leq \mathfrak{h} \leq \mathfrak{c}$ and that ZFC does not determine the exact value of $\mathfrak{h}$. For example, Martin's Axiom implies $\mathfrak{h} = \mathfrak{c}$; on the other hand, $\mathfrak{h} = \aleph_1$ holds in the Cohen model for the failure of the Continuum Hypothesis. As $\{0,1\}^{\omega}/\mathrm{Fin}$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$ it follows that $\{0,1\}^{\omega}/\mathrm{Fin}$ corresponds to the Stone-Čech remainder, ω^{*}, of the compactification of the naturals. The study of the distributivity for this space was initiated in [2]. $(\mathcal{P}(\omega))^{\omega}/\operatorname{Fin}$ topologically corresponds to $(\beta\omega \times \omega)^{*}$. The topological correspondent of $(\mathcal{P}(\omega)/\mathrm{fin})^{\omega}/\operatorname{Fin}$ is $(\omega \times \omega^{*})^{*}$ and one of the first papers studying the distributivity of this space is [7] where this space is denoted by ω^{2*}. Finally, one can choose to work with, $\mathbb{A}$, the Boolean algebra of clopen subsets of the Cantor set 2^{ω} as the representative of the atomless countable Boolean algebra; then one can see that $\mathbb{A}^{\omega}/\operatorname{Fin}$ is isomorphic to the algebra of clopen subsets of $\beta(2^{\omega} \times \omega) \setminus (2^{\omega} \times \omega)$. This space is, in particular, co-absolute with $\beta\mathbb{R} \setminus \mathbb{R}$. The study of the distributivity number of $\beta\mathbb{R} \setminus \mathbb{R}$ was initiated in [8].

2. Computing $\mathfrak{h}(\mathbb{B}^{\omega}/\operatorname{Fin})$.

Our terminology and notation are mostly standard and follows that of [12] and [3]. We refer the reader to those sources for undefined notions here. The phrase "*for almost all*" will mean "for all but, possibly, finitely many of".

Since $\mathcal{P}(\omega)/\mathrm{fin}$ is regularly embedded in $\mathbb{B}^{\omega}/\operatorname{Fin}$ for any Boolean algebra $\mathbb{B}$. In [1] the authors showed that $\mathbb{B}^{\omega}/\operatorname{Fin}$ can be written as an iteration of $\mathcal{P}(\omega)/\mathrm{fin}$ and an ultra-power of $\mathbb{B}$ modulo $\mathcal{U}$. For the sake of completeness we present here their result together with their short proof.

Proposition 2.1 ([1]). *$\mathbb{B}^{\omega}/\operatorname{Fin}$ is forcing equivalent to the iteration*

$$\mathcal{P}(\omega)/\mathrm{fin} * \mathbb{B}^{\omega}/\dot{\mathcal{U}},$$

where $\dot{\mathcal{U}}$ is the $\mathcal{P}(\omega)/\mathrm{fin}$-name for the Ramsey ultrafilter added by $\mathcal{P}(\omega)/\mathrm{fin}$.

Proof. Define a function $\Phi : \mathbb{B}^{\omega}/\operatorname{Fin} \to \mathcal{P}(\omega)/\mathrm{fin} * \mathbb{B}^{\omega}/\dot{\mathcal{U}}$ by putting $\Phi(f) = \left\langle \operatorname{supp}(f), [\dot{f}]_{\mathcal{U}} \right\rangle$, where $[\dot{f}]_{\mathcal{U}}$ is a $\mathcal{P}(\omega)/\mathrm{fin}$-name for

$$\{g \in \mathbb{B}^{\omega} : \{n \in \omega : f(n) = g(n)\} \in \mathcal{U}\}.$$

It is easy to verify that Φ is a dense embedding. $\square$

A consequence of the regular embedding of $\mathcal{P}(\omega)/\mathrm{fin}$ into $\mathbb{B}^\omega/\operatorname{Fin}$ is that

$$\mathfrak{h}\left(\mathbb{B}^\omega/\operatorname{Fin}\right) \leq \mathfrak{h} \tag{1}$$

for any Boolean algebra $\mathbb{B}$. As we said before, for $\mathbb{B} = \{0,1\}$ ZFC does not determines the value of $\mathfrak{h}$. One more comment we can make about this is that the natural forcing to increase $\mathfrak{h}$ is the Mathias forcing; thus in the Mathias model $\mathfrak{h}$ is $\aleph_2$.

For $\mathbb{B} = \mathbb{A}$, the best known result is in [1]; it is a nice theorem which improves the result in [8] which says that $\mathfrak{h}\left(\mathbb{A}^\omega/\mathrm{Fin}\right) = \aleph_1$ in the Mathias model.

THEOREM 2.2 ([1]). $\mathfrak{h}\left(\mathbb{A}^\omega/\operatorname{Fin}\right) \leq \min\left\{\mathfrak{h},\mathsf{add}\left(\mathcal{M}\right)\right\}.$

In [11] we use a natural modification of Mathias forcing which increases $\mathfrak{h}\left(\mathbb{A}^\omega/\operatorname{Fin}\right)$ the same way that Mathias forcing increases $\mathfrak{h}$; that is, we produce a model where there is a tree π-base for $\mathbb{A}^\omega/\operatorname{Fin}$ of height ω_2 without branches of length ω_2. A *tree π-base* for a space X is a dense subset of the regular open algebra of subsets of X which forms a tree when ordered by reverse inclusion.

The forcing used in [11] uses a lot of the topological structure of the reals but in the general case it can be defined as follows: $\mathbb{M}_{\mathbb{B}}$ is the forcing whose conditions are pairs $\langle s, B\rangle$ where s is a finite subset of $\mathbb{B}^+$ and B is a regular open subset of $\mathbb{B}$ with $s \cap B = \emptyset$ and with the ordering $\langle s, B\rangle \leq \langle r, A\rangle$ if and only if $r \subseteq s \subseteq r \cup A$ and $B \subseteq A$. Recall that $B \subseteq \mathbb{B}$ is regular open if whenever $a \leq b$ and $b \in B$ we have $a \in B$, and for every $b \notin B$ there is $a \leq b$ such that $B_a \cap B = \emptyset$, where $B_a = \{x \in \mathbb{B} : x \leq a\}$.

The first computation we do is for $\mathcal{P}(\omega)^\omega/\mathrm{Fin}$. We wish to thank Professor Jörg Brendle for his help to fix a previous proof. This algebra is isomorphic to the algebra $\mathcal{P}(\omega)/\mathrm{fin}\times\mathcal{P}(\omega)$.

PROPOSITION 2.3. $\mathfrak{h}\left(\mathcal{P}(\omega)^\omega/\mathrm{Fin}\right) = \mathfrak{h}$.

Proof. For the purpose of the proof, for a function $f : A \to \omega$ and $A \subseteq \omega$ denote by A^f the set $\{\langle n, f(n)\rangle : n \in A\}$. Then it is easy to see that the family $\mathcal{D} = \left\{A^f : A \in [\omega]^\omega, f \in \omega^A\right\}$ is a dense subset of $\mathcal{P}(\omega)^\omega/\mathrm{Fin}$.

It follows that $\mathfrak{h}\left(\mathcal{P}(\omega)^\omega/\mathrm{Fin}\right) \leq \mathfrak{h}$ by (1). To prove the other inequality let $\kappa < \mathfrak{h}$ and consider a family $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ of maximal antichains in $\mathcal{D}$. Given $A^f \in \mathcal{A}_0$, let $\mathcal{C}_{\alpha,f}$ be a maximal antichain in $\mathcal{P}(\omega)^\omega/\mathrm{Fin}$ below A^f and below $\mathcal{A}_\alpha$. Fix a maximal almost disjoint family $\mathcal{B}_{\kappa,f} = \left\{B \subseteq \omega : B^f \in \mathcal{C}_{\alpha,f}\right\}$ on A. Since $\kappa < \mathfrak{h}$ there is $\mathcal{B}_{\kappa,f}$ which is a common refinement of the families $\mathcal{B}_{\alpha,f}$ for $\alpha < \kappa$.

Letting $\mathcal{A}_\kappa = \left\{B^{f\restriction B} : B \in \mathcal{B}_{\kappa,f} \;\&\; f \in \mathcal{A}_0\right\}$ we obtain a common refinement for each $\mathcal{A}_\alpha$, as we wanted to show. $\square$

We pass now to compute $\mathfrak{h}\left(\left(\mathcal{P}(\omega)/\mathrm{fin}\right)^{\omega}/\,\mathrm{Fin}\right)$; for short we write $\mathfrak{h}\left(\omega^{2*}\right)$, see the introduction. Dow showed that a tree π-base for ω^{2*} cannot be ω_2-closed and that Martin's Axiom (actually $\mathfrak{p}=\mathfrak{c}$) implies that the boolean algebra $\left(\mathcal{P}(\omega)/\mathrm{fin}\right)^{\omega}/\,\mathrm{Fin}$ (which by the way is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}\times\mathrm{fin}$) is $\mathfrak{c}$-distributive, and hence $\mathfrak{h}\left(\omega^{2*}\right)=\mathfrak{c}$. We are showing now that exact value of $\mathfrak{h}\left(\omega^{2*}\right)$ cannot be decided. At first glance one would think that $\mathfrak{h}\left(\omega^{2*}\right)=\mathfrak{h}$; however in the Mathias model they differ. To show that we are going to use a game theoretical characterization of $\mathfrak{h}(\mathbb{B})$. For more on games and distributivity laws in Boolean algebras see [6].

Let us consider the following game first introduced in [10]. For a homogeneous Boolean algebra $\mathbb{B}$ and for any ordinal α, $\mathsf{G}(\mathbb{B},\alpha)$ is the game of length α between Player I and Player II, who alternatively choose non-zero elements $b^{I}_{\beta},\ b^{II}_{\beta}\in\mathbb{B}$ for $\beta<\alpha$ such that for $\beta<\beta'<\alpha$:

$$b^{I}_{\beta}\geq b^{II}_{\beta}\geq b^{I}_{\beta'}\geq b^{II}_{\beta'}.$$

In the end, Player II wins if and only if the sequence of moves has no lower bound (this might happen if at some step $\beta<\alpha$, Player I does not have a legal move).

Lemma 2.4. *$\mathfrak{h}(\mathbb{B})$ is the minimum cardinal κ such that in the game $\mathsf{G}(\mathbb{B},\kappa)$ Player II has a winning strategy.*

The main result in [13] follows from the next two propositions which are going to be used in the sequel. We introduce some notation needed. Firstly, S^2_1 is the set of all ordinals $\alpha<\omega_2$ with $\mathrm{cf}(\alpha)=\omega_1$; while $\mathbb{P}_\beta$ denotes the countable support iteration of length $\beta\leq\omega_2$ of Mathias forcing, $\mathbb{M}$, and $\dot{G}_\alpha$ denotes the $\mathbb{P}_\alpha$-name for the $\mathbb{P}_\alpha$-generic filter. Also, the quotient forcing $\mathbb{P}_{\omega_2}/\dot{G}_\alpha$ is denoted by $\mathbb{P}_{\alpha\omega_2}$. Recall that ultrafilters $\mathcal{U}_0$ and $\mathcal{U}_1$ are *Rudin-Keisler equivalent* if exists a bijection $f:\omega\to\omega$ such that $\mathcal{U}_1=\{f[U]:U\in\mathcal{U}_0\}$. An ultrafilter $\mathcal{R}$ is a *Ramsey ultrafilter* if for every $k,n\in\omega$ and every partition $\varrho:[\omega]^n\to k$ there exists $H\in\mathcal{R}$ homogeneous for ϱ; that is, $\varrho\restriction[H]^n$ is constant. Ramsey ultrafilters are also known as selective ultrafilters. See [12, p. 478] and [3, p. 235] for more on Ramsey ultrafilters.

Proposition 2.5 ([13]). *There exists an ω_1-club $C\subseteq S^2_1$ such that for every $\alpha\in C$ the following holds: If $\dot{r}$ is a $\mathbb{P}_{\alpha\omega_2}$-name such that $\mathbb{P}_{\alpha\omega_2}\Vdash$ "$\dot{r}$ induces a Ramsey ultrafilter on $\left([\omega]^{\omega}\right)^{V[\dot{G}_\alpha]}$", then there is a $\mathbb{P}_{\alpha\omega_2}$-name $\dot{r}'$ such that $\mathbb{P}_{\alpha\omega_2}\Vdash$ "$\dot{r}'\in V\left[\dot{G}_{\alpha+1}\right]$, $\dot{r}$ and $\dot{r}'$ generate the same ultrafilter on $\left([\omega]^{\omega}\right)^{V[\dot{G}_\alpha]}$".*

Proposition 2.6 ([13]). *Suppose that V is a model of CH and that $\dot{r}$ is a $\mathbb{M}$-name such that $\mathbb{M}\Vdash$ "$\dot{r}$ induces a Ramsey ultrafilter $\dot{\mathcal{R}}$ on $\left([\omega]^{\omega}\right)^{V}$". Then $\mathbb{M}\Vdash$ "$\dot{\mathcal{U}}$ and $\dot{\mathcal{R}}$ are Rudin-Keisler equivalent by some function $f\in\left(\omega^{\omega}\right)^{V}$", where $\mathcal{U}$ is the Ramsey ultrafilter added by $\mathcal{P}(\omega)/\mathrm{fin}$.*

THEOREM 2.7. *Assume V is a model of CH. If G is $\mathbb{P}_{\omega_2}$ -generic over V, then $V[G] \vDash \mathfrak{h}\left(\omega^{2*}\right) = \aleph_1$.*

Proof. Suffices to define a winning strategy for Player II in the game

$$\mathsf{G}\left(\left(\mathcal{P}(\omega)/\mathrm{fin}\right)^{\omega}/\,\mathrm{Fin}, \omega_1\right)$$

played in $V[G]$. In order to do that, fix a ω_1-club $C \subseteq S_1^2$ as in Proposition 2.5. For every $x \in V[G]$, let $o(x) = \min\{\alpha < \omega_2 : x \in V[G_\alpha]\}$ and fix a $\Gamma : \omega_1 \to \omega_1 \times \omega_1$ bijection such that $\Gamma(\alpha) = \langle \beta, \delta \rangle$ implies $\beta \leq \alpha$. Since $V[G_\alpha] \vDash$ CH, for each $\alpha < \omega_2$, there is a function $g_\alpha : \omega_1 \to V[G_\alpha]$ which enumerates all triples $\langle a, \varrho, f \rangle \in V[G_\alpha]$ such that $a \in [\omega]^\omega$, $\varrho : [\omega]^n \to k$ for some $k, n \in \omega$ and $f : \omega \to \omega$ is a function.

The winning strategy for Player II is as follows:

If $\left\langle \left\langle p_\xi^I, p_\xi^{II} \right\rangle : \xi < \omega_1 \right\rangle$ is a play, there is $\alpha \in C$ such that $\left\langle p_\xi^{II}(n) : \xi < \omega_1 \right\rangle$ generates Ramsey ultrafilters on $\left([\omega]^\omega\right)^{V[G_\alpha]}$ for each $n \in \omega$ such that any two of them are not Rudin-Keisler equivalent by any $f \in \left(\omega^\omega\right)^{V[G_a]}$.

The α-th move of Player II in a given play $\left\langle \left\langle p_\xi^I, p_\xi^{II} \right\rangle : \xi < \omega_1 \right\rangle$ is in such a way that if $\Gamma(\alpha) = \langle \beta, \delta \rangle$, $\xi \in C$ is minimal with the property that $\xi \geq \sup\left\{o\left(p_\eta^I(n) : \eta < \beta \;\&\; n \in \omega\right)\right\}$, and $g_\xi(\delta) = \langle a, \varrho, f \rangle$, then

1. $p_\alpha^{II}(n) \subseteq^* p_\alpha^I(n)$ for almost all $n \in \omega$,
2. $p_\alpha^{II}(n) \subseteq a$ or $p_\alpha^{II}(n) \cap a = \emptyset$,
3. $p_\alpha^{II}(n)$ is ϱ-homogeneous,
4. $f\left[p_\alpha^{II}(n)\right] \cap p_\alpha^{II}(m) =^* \emptyset$, for all $m, n \in \omega$.

To see that this is possible suppose we have chosen $p_\alpha^{II}(k)$ for $k < n$ satisfying (1), (2), (3) and (4) for $i, j < n$:

$$f\left[p_\alpha^{II}(i)\right] \cap p_\alpha^{II}(j) =^* \emptyset.$$

To choose $p_\alpha^{II}(n)$ start by choosing some $B_n^n \subseteq p_\alpha^I(n)$ which is ϱ-homogeneous and either $B_n^n \subseteq a$ or $B_n^n \cap a = \emptyset$. Then we keep choosing sets B_m^n for $m > n$ as follows: Assuming B_m^n has been defined, let B_{m+1}^n be B_m^n if $f[B_m^n] \cap p_\alpha^I(m+1) =^* \emptyset$, otherwise let B_{m+1}^n be some infinite subset of B_m^n such that $p_\alpha^I(m+1) \setminus f\left[B_{m+1}^n\right] \neq^* \emptyset$ and shrink $p_\alpha^I(m+1)$ to become $p_\alpha^I(m+1) \setminus f\left[B_{m+1}^n\right]$. (Here we abuse of the notation and we call this new set again $p_\alpha^I(m+1)$.) Finally let B be some infinite $B \subseteq^* B_m^n$ for all $m \geq n$.

Since the set $f[B]$ is almost disjoint from each $p_\alpha^I(m)$ for $m > n$ and the new sets $p_\alpha^{II}(m)$ are going to be subsets of $p_\alpha^I(m)$ the clause (4) will be preserved if we let $p_\alpha^{II}(n)$ be any infinite subset of B.

Notice that the fact that C is an ω_1-club implies that the strategy is as desired.

To finish the proof we show that this strategy is a winning strategy for Player II. Suppose that $\langle p_\beta : \beta < \omega_1 \rangle$ are the moves of Player II according to the strategy, and suppose that the game is won by Player I. Then, there exists $r \in V[G]$ such that $r(n) \in [\omega]^\omega$ for almost all $n \in \omega$ and $r(n) \subseteq^* p_\beta(n)$ for almost all $n \in \omega$ and all $\beta < \omega_1$. Fix $\alpha \in C$ and Ramsey ultrafilters $\mathcal{U}(n)$ on $([\omega]^\omega)^{V[G_\alpha]}$ for $n < \omega$ such that each $\mathcal{U}(n)$ is generated by $\langle p_\beta(n) : \beta < \omega_1 \rangle$ and no two of them are Rudin-Keisler equivalent for any $f \in \omega^\omega \cap V[G_\alpha]$. Then $\mathcal{U}(n)$ is generated by $r(n)$. By Proposition 2.5, $r \in V[G_{\alpha+1}]$ and by Proposition 2.6 $\mathcal{U}(n)$ is Rudin-Keisler equivalent to $\mathcal{U}$ by functions in $\omega^\omega \cap V[G_\alpha]$. However, by construction this is impossible. □

3. Final remarks

The results presented here can be the beginning of a whole research on the cardinal invariants of algebras of the type $\mathbb{B}/\mathcal{I}$ where $\mathbb{B}$ is a subalgebra of $\mathcal{P}(\omega)$ and $\mathcal{I}$ is an ideal over the natural numbers. As an instance of this, recall that by a result of Mazur an ideal $\mathcal{I}$ is an F_σ ideal if and only if it is equal to $\mathrm{Fin}(\varphi) = \{I \subseteq \omega : \varphi(I) < \infty\}$, for some lower semicontinuous submeasure φ. This can be used to easily show that $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed and hence $\mathfrak{h}_\mathcal{I} = \mathfrak{h}(\mathcal{P}(\omega)/\mathcal{I}) > \aleph_0$. We would like to know how to compute $\mathfrak{h}_\mathcal{I}$ for F_σ ideals $\mathcal{I}$.

The base tree matix lemma of Balcar, Pelant and Simon [2] have proved to be an important tool, so we ask:

Problem 3.1: For which ideals is the base tree matrix lemma still true for $\mathcal{P}(\omega)/\mathcal{I}$?

Problem 3.2: Does the base tree matrix lemma implies that the collapse of $\mathfrak{c}$ to the respective $\mathfrak{h}$?

Problem 3.3: What is the relationship between $\mathfrak{h}$ and $\mathfrak{h}_\mathcal{I}$ for F_σ ideals $\mathcal{I}$?

Going back to $\mathcal{P}(\omega)^\omega/\mathrm{Fin}$, observe that if $\mathcal{A}$ is a maximal almost disjoint family of subsets of ω and for each $A \in \mathcal{A}$ we define $f_A \in \mathcal{P}(\omega)^\omega$ by

$$f_A(n) = \begin{cases} \omega, & \text{if } n \in A \\ \emptyset, & \text{if } n \notin \emptyset. \end{cases}$$

Then $\{f_A : A \in \mathcal{A}\}$ is a maximal antichain in $\mathcal{P}(\omega)^\omega/\mathrm{Fin}$. It follows that $\mathfrak{a}(\mathcal{P}(\omega)^\omega/\mathrm{Fin}) \leq \mathfrak{a}$.

Problem 3.4: Does $\mathfrak{a} \leq \mathfrak{a}(\mathcal{P}(\omega)^\omega/\mathrm{Fin})$?

PROBLEM 3.5: Does $\mathfrak{b} \leq \mathfrak{a}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right)$?

Similar arguments to the above one shows that

$$\mathfrak{p}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right) \leq \mathfrak{p}, \mathfrak{t}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right) \leq \mathfrak{t} \text{ and } \mathfrak{s}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right) \leq \mathfrak{s}.$$

PROBLEM 3.6: Does $\mathfrak{t}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right) \geq \mathfrak{t}$?

PROBLEM 3.7: Does $\mathfrak{s}\left(\mathcal{P}(\omega)^{\omega}/\mathrm{Fin}\right) \geq \mathfrak{s}$?

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