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$\mathfrak{c}$-Many types of a Ψ-space

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ABSTRACT

We show that there are $\mathfrak{c}$ many AD families of the same (uncountable) size whose Ψ-spaces are pairwise non-homeomorphic and they can be Luzin families or branch families of 2^ω.

1. Introduction

An *almost disjoint family* of subsets of the natural numbers ω (or any other countable set) is a family of infinite subsets of ω so that any two different elements of the family have finite intersection. If $\mathcal{A}$ is an almost disjoint family (AD family, for short) on ω, define the topological space $\Psi(\mathcal{A}) = (\omega \cup \mathcal{A}, \tau)$ as follows: ω is a discrete subset of $\Psi(\mathcal{A})$; basic neighborhoods of a point $x \in \mathcal{A}$ are of the form $\{x\} \cup (x \setminus F)$, where $F \subseteq \omega$ is finite.

Ψ-spaces have been well studied throughout the years because they are candidates to give examples or counterexamples of many topological concepts. There are nice properties Ψ-spaces satisfy: they are Haus-

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dorff, separable, first countable, locally compact and zero dimensional. For topological and combinatorial aspects of Ψ-spaces see [1] and [2], respectively.

Daniel Bernal-Santos and Salvador García-Ferreira wondered if $C_p(\Psi(\mathcal{A}))$ and $C_p(\Psi(\mathcal{B}))$ are homeomorphic whenever $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic as subspaces (considered as sets of characteristic functions) of the Cantor set 2^ω with the usual topology. To understand the space $C_p(\Psi(\mathcal{A}))$ better, in personal communications they asked us for a more elementary question:

Question 1 *(Bernal-Santos, García-Ferreira). If $X, Y \subseteq 2^\omega$ are homeomorphic, are $\Psi(\mathcal{A}_X)$ and $\Psi(\mathcal{A}_Y)$ homeomorphic?*

Here, $\mathcal{A}_X := \{\{x \restriction n : n \in \omega\} : x \in X\}$ is the almost disjoint family of branches determined by the elements of X. It is well known that under $\mathsf{MA} + \neg\mathsf{CH}$, every set $X \subseteq 2^\omega$ of size less than the continuum is a Q-set (recall that a separable metrizable space X is a Q-set if every subset of X is G_δ in X), and thus, $\Psi(\mathcal{A}_X)$ is normal. Like this, there are many topological properties of $X \subseteq 2^\omega$ that have an effect on the Ψ-space $\Psi(\mathcal{A}_X)$. One might think that $\mathsf{MA} + \neg\mathsf{CH}$ is a good ingredient to conjecture that the answer is affirmative. However, we answer Question 1 negatively since Theorem 14 shows that in ZFC there are different types of spaces $\Psi(\mathcal{A}_X), \Psi(\mathcal{A}_Y)$ even when X and Y are homeomorphic.

Recall that an AD family $\mathcal{A}$ is *Luzin* if it can be enumerated as

$$\mathcal{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$$

in such way that $\forall \alpha < \omega_1 \; \forall n \in \omega \;\; (|\{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}| < \omega)$. Branch and Luzin families are in some sense "orthogonal", precisely because the normality of their Ψ-spaces might hold in the former and breaks down badly in the latter. We show in Theorem 13 that in ZFC there are different types of Ψ-spaces for Luzin families.

Focusing on AD families of size ω_1, Michael Hrušák formulated the following question in a local seminar:

Question 2 *(Hrušák). Is it consistent that there is an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A}) \simeq \Psi(\mathcal{B})$, whenever $\mathcal{B} \subseteq \mathcal{A}$ and $|\mathcal{A}| = |\mathcal{B}|$?*

Observe that $2^\omega < 2^{\omega_1}$ (in particular CH) implies that the answer to Question 2 is negative by the simple fact that given an AD family $\mathcal{A}$ of size ω_1, there are only $\mathfrak{c}$ many subspaces $\Psi(\mathcal{B})$ for which $\Psi(\mathcal{A}) \simeq \Psi(\mathcal{B})$ (there are only $\mathfrak{c}$ permutations of ω), and there are 2^{ω_1} many subsets of $\mathcal{A}$ of size ω_1. We believe that it is a very interesting question; we conjecture that the answer is no, but our methods do not work to solve it.

2. Basic facts

Our notation is standard and follows closely [1] and [2]. Similarly, we use $f(A)$ to denote the evaluation of the function f at the point A in its domain while $f[A]$ denotes the image of the set A under the function f. For sets A and B, we say that $A \subseteq^* B$, in words that A is almost contained in B, if $A \setminus B$ is a finite set. Likewise, $A =^* B$ if and only if $A \subseteq^* B$ and $B \subseteq^* A$. For a set Z and a cardinal κ, denote by $[Z]^\kappa$, $[Z]^{<\kappa}$ and $[Z]^{\leq\kappa}$ the families of all subsets of Z of size κ, less than κ and less than or equal to κ, respectively. If $x \in 2^\omega$, we denote

$$\widehat{x \downarrow n} = \{x \restriction k \in 2^{<\omega} : n \leq k\} \quad \text{and} \quad \widehat{x} := \widehat{x \downarrow 0}.$$

The families $\mathcal{A}_X$ defined above, where $X \subseteq 2^\omega$, are canonical AD families on $2^{<\omega}$, and there are of any size below the continuum. Under a bijection between ω and $2^{<\omega}$ we can consider $\Psi(\mathcal{A}_X)$. Perhaps the families

$\mathcal{A}_X$ were first studied by F. Tall [3] when he showed that if $X \subseteq 2^\omega$, then X is a Q-set if and only if $\Psi(\mathcal{A}_X)$ is normal.

The following lemma shows how a homeomorphism between Ψ-spaces looks like.

Lemma 3. *Let $\mathcal{A}, \mathcal{B}$ be almost disjoint families on ω and $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ be bijective. Then, H is a homeomorphism if and only if $H[\omega] = \omega$ and for every $x \in \mathcal{A}$, $H[x]$ and $H(x)$, as subsets of ω, are almost equal.*

Proof. $\Rightarrow$) Since H is bijective and must send isolated points to isolated points, it is clear that $H[\omega] = \omega$, that is, H is a permutation on ω. Now, let $x \in \mathcal{A}$ and $\{H(x)\} \cup (H(x) \setminus F)$ be a neighborhood of $H(x)$, where $F \in [\omega]^{<\omega}$. By continuity, there is $F' \in [\omega]^{<\omega}$ such that $H[\{x\} \cup (x \setminus F')] \subseteq \{H(x)\} \cup (H(x) \setminus F)$. Notice that the former set $H[\{x\} \cup (x \setminus F')]$ is the set $\{H(x)\} \cup H[x \setminus F']$. Then $H[x \setminus F'] \subseteq H(x) \setminus F$, and thus, $H[x] \subseteq^* H(x)$. Use the fact that H is open and similar arguments to get $H[x] \supseteq^* H(x)$.

$\Leftarrow$) We will see that H is continuous; to see that H is open use similar arguments as above. Let $x \in \mathcal{A}$ and $\{H(x)\} \cup (H(x) \setminus F)$ be a neighborhood of $H(x)$, where $F \in [\omega]^{<\omega}$. Since $H[x] =^* H(x)$, there is $F' \in [\omega]^{<\omega}$ such that $H[x] \setminus H[F'] \subseteq H(x) \setminus F$. Since H is a permutation on ω, we have $H[x \setminus F'] = H[x] \setminus H[F']$, and then, $H[\{x\} \cup (x \setminus F')] \subseteq \{H(x)\} \cup (H(x) \setminus F)$. □

For $s \in 2^{<\omega}$, let $\langle s \rangle = \{t \in 2^{<\omega} : s \subseteq t\}$ and $[\langle s \rangle] = \{x \in 2^\omega : s \subseteq x\}$.

Lemma 4. *Let $X \subseteq 2^\omega$ be a set of size κ, $cf(\kappa) > \omega$. Then there are infinitely many $n \in \omega$ for which there are different elements $s, t \in 2^n$ such that $|[\langle s \rangle] \cap X| = \kappa = |[\langle t \rangle] \cap X|$.*

Proof. Suppose for a contradiction that for every $n \in \omega$ there is a unique $s_n \in 2^n$ such that $X_n := [\langle s_n \rangle] \cap X$ has size κ. Let $Y_n = X \setminus X_n$. Notice that $|Y_n| < \kappa$, and since $cf(\kappa) > \omega$, $Y = \bigcup_{n\in\omega} Y_n$ has size less than κ. This is a contradiction because $X \setminus Y = \bigcap_{n\in\omega} X_n$ has size κ and it is contained in the set $\bigcap_{n\in\omega}[\langle s_n \rangle]$ that has at most one element. □

Notice that by the previous Lemma, one can actually get infinitely many $n \in \omega$ for which there is $s \in 2^n$ such that $|[\langle s^\frown 0 \rangle] \cap X| = \kappa = |[\langle s^\frown 1 \rangle] \cap X|$. For an AD family $\mathcal{A}$ on ω, we obtain the next observation by considering $\{\chi_A : A \in \mathcal{A}\} \subseteq 2^\omega$, where χ_A is the characteristic function of A.

Remark 5. Let $\mathcal{A}$ be an *AD* family of size κ with $cf(\kappa) > \omega$. Then

$$\forall n \in \omega\ \exists m > n(|\{x \in \mathcal{A} : m \in x\}| = |\{x \in \mathcal{A} : m \notin x\}| = \kappa).$$

Lemma 6. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ with $cf(\kappa) > \omega$ and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. Then for all $n \in \omega$ there are $x, y, z \in \mathcal{A}$ such that*

1. $\max\{x \cap y\} > n \ \wedge\ x \cap y \subsetneq x \cap z$; *and*
2. $\max\{h(x) \cap h(y)\} > n \ \wedge\ h(x) \cap h(y) \subsetneq h(x) \cap h(z)$.

Proof. Fix $n \in \omega$. By Remark 5, choose $m > n$ and $\mathcal{A}_0 \in [\mathcal{A}]^\kappa$ such that for every $x \in \mathcal{A}_0$, $m \in x$ and $m \in h(x)$. Now, fix $y \in \mathcal{A}_0$ and apply Pigeonhole principle to the set $\{x \cap y : x \in \mathcal{A}_0 \wedge x \neq y\}$. There are $F_0 \in [\omega]^{<\omega}$ and $\mathcal{A}_1 \in [\mathcal{A}_0]^\kappa$ such that for all $x \in \mathcal{A}_1$, $x \cap y = F_0$. There are also $G_0 \in [\omega]^{<\omega}$ and $\mathcal{B}_1 \in [h[\mathcal{A}_1]]^\kappa$ such that for all $w \in \mathcal{B}_1$, $w \cap h(y) = G_0$. Let $\mathcal{A}_2 = h^{-1}[\mathcal{B}_1]$.

At this point we have that for any $\{x, z\} \in [\mathcal{A}_2]^2$, $F_0 = x \cap y = z \cap y$ and $G_0 = h(x) \cap h(y) = h(z) \cap h(y)$, simultaneously. This already implies that $x \cap y \subseteq x \cap z$ and $h(x) \cap h(y) \subseteq h(x) \cap h(z)$. To find elements so that the inclusions are strictly proper, since $|\mathcal{A}_2| = \kappa$, use again Remark 5 to get $m' > max\ F_0 \cup G_0 \cup \{m\}$

and $\mathcal{A}_3 \in [\mathcal{A}_2]^\kappa$ such that for any $x \in \mathcal{A}_3$, $m' \in x$ and $m' \in h(x)$. Now, if $\{x, z\} \in [\mathcal{A}_3]^2$, then $x \cap y = F_0 \subsetneq F_0 \cup \{m'\} \subseteq x \cap z$ and $h(x) \cap h(y) = G_0 \subsetneq G_0 \cup \{m'\} \subseteq h(x) \cap h(z)$. □

Definition 7. Let $\mathcal{A}, \mathcal{B}$ be AD families on ω of size κ and $h : \mathcal{A} \to \mathcal{B}$ be bijective. We say that h is of *dense oscillation* if for each $\mathcal{A}' \in [\mathcal{A}]^\kappa$ there are $x, y, z \in \mathcal{A}'$ such that $|x \cap z \setminus x \cap y| \neq |h(x) \cap h(z) \setminus h(x) \cap h(y)|$.

Proposition 8. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ with $cf(\kappa) > \omega$ and $h : \mathcal{A} \to \mathcal{B}$ be of dense oscillation. Then, there is no homeomorphism from $\Psi(\mathcal{A})$ to $\Psi(\mathcal{B})$ that extends h.*

Proof. Suppose for a contradiction that $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ is a homeomorphism extending h. By Lemma 3, for every $A \in \mathcal{A}$, $H[A] =^* H(A)$. So, for $A \in \mathcal{A}$, consider the finite sets $F_A = \{n \in A : H(n) \notin H(A)\}$ and $G_A = \{n \in H(A) : H^{-1}(n) \notin A\}$.

There are $\mathcal{A}' \in [\mathcal{A}]^\kappa$ and $F, G \in [\omega]^{<\omega}$ such that for all $A \in \mathcal{A}'$, $F = F_A$ and $G = G_A$. If $x, y, z \in \mathcal{A}'$ are different, then

$$(x \cap z \setminus x \cap y) \cap F = \emptyset \quad \text{and} \quad \Big((H(x) \cap H(z)) \setminus (H(x) \cap H(y)) \Big) \cap G = \emptyset.$$

Moreover, $m \in x \setminus F$ implies $H(m) \in H(x)$, and $H(m) \in H(x) \setminus G$ implies $m \in x$. From this, one can deduce that

$$|x \cap z \setminus x \cap y| = |H(x) \cap H(z) \setminus H(x) \cap H(y)|,$$

contradicting the dense oscillation property of $H \restriction \mathcal{A} = h$. □

Definition 9. Let $A, B \subseteq \omega$.

- A and B are *oscillating* if

$$\forall \{x, y\} \in [A]^2 \ \forall \{w, z\} \in [B]^2 \ (|y - x| \neq |z - w|).$$

- A and B are *almost oscillating* if there is $n \in \omega$ such that $A \setminus n$ and $B \setminus n$ are oscillating.

Proposition 10. *There are $\mathfrak{c}$ many infinite subsets of ω pairwise almost oscillating.*

Proof. From ω, we first construct two oscillating sets $A = \bigcup_{n \in \omega} A_n, B = \bigcup_{n \in \omega} B_n$. Fix $A_0 = \{0\}, B_0 = \{1\}$. Suppose constructed $A_n = \{a_0, \ldots, a_n\}, B_n = \{b_0, \ldots, b_n\}$ oscillating. Let $a_{n+1} \in \omega$ such that $a_{n+1} - a_n > b_n - b_0$ and $b_{n+1} \in \omega$ such that $b_{n+1} - b_n > a_{n+1} - a_0$. Observe that $A_{n+1} = A_n \cup \{a_{n+1}\}, B_{n+1} = B_n \cup \{b_{n+1}\}$ are oscillating as well as will be A and B.

Notice that the construction is hereditary. That is, for any $X \in [\omega]^\omega$, there are $A, B \in [X]^\omega$ oscillating. This allows to define a Cantor tree induced by these partitions. Each branch of the Cantor set, $f \in 2^\omega$, represents a decreasing sequence $\langle A_{f \restriction n} : n \in \omega \rangle$ of infinite sets of ω such that for any other branch $g \in 2^\omega$, we have that $A_{f \restriction k}, A_{g \restriction l}$ are oscillating whenever $k, l > \Delta(f, g)$. Now, for every sequence $\langle A_{f \restriction n} : n \in \omega \rangle$, consider a pseudointersection P_f of $\{A_{f \restriction n} : n \in \omega\}$. Observe that for any two sequences $\langle A_{f \restriction n} : n \in \omega \rangle$, $\langle A_{g \restriction n} : n \in \omega \rangle$, their pseudointersections P_f, P_g are almost oscillating. □

Corollary 11. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ, with $cf(\kappa) > \omega$, and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. If $A = \{|x \cap y| : \{x, y\} \in [\mathcal{A}]^2\}$ and $B = \{|x \cap y| : \{x, y\} \in [\mathcal{B}]^2\}$ are almost oscillating, then there is $\mathcal{A}' \in [\mathcal{A}]^\kappa$ such that $h \restriction \mathcal{A}'$ is of dense oscillation.*

Proof. Let $n \in \omega$ such that $A \setminus n$ and $B \setminus n$ are oscillating. Iterating $n+1$-many steps Remark 5, we can find a subfamily $\mathcal{A}_0 \in [\mathcal{A}]^\kappa$ such that for any $\{x,y\} \in [\mathcal{A}_0]^2$, $|x \cap y| \geq n+1$. In the same way, we can find a subfamily $\mathcal{B}_1 \in [h[\mathcal{A}_0]]^\kappa$ such that for any $\{w,z\} \in [\mathcal{B}_1]^2$, $|w \cap z| \geq n+1$. Then, $\mathcal{A}_1 := h^{-1}[\mathcal{B}_1] \in [\mathcal{A}_0]^\kappa$. Notice that for any $\{x,y\} \in [\mathcal{A}_1]^2$,

$$n+1 \leq min\{|x \cap y|, |h(x) \cap h(y)|\}. \tag{1}$$

We do this to avoid the possibility to obtain an intersection of size at most n in order to reach "an oscillation".

To see that $\mathcal{A}' := \mathcal{A}_1$ is the desired family, choose $\mathcal{A}'' \in [\mathcal{A}']^\kappa$. Apply Lemma 6 to n and $h \restriction \mathcal{A}'' : \mathcal{A}'' \to h[\mathcal{A}'']$, and get $x, y, z \in \mathcal{A}''$ such that $x \cap y \subsetneq x \cap z$, $h(x) \cap h(y) \subsetneq h(x) \cap h(z)$ and $n < min\{max\{x \cap y\}, max\{h(x) \cap h(y)\}\}$ (observe that this last inequality was implied by (1)). Thus, there are $\{a_0, a_1\} \in [A \setminus n]^2$ and $\{b_0, b_1\} \in [B \setminus n]^2$ such that $|x \cap z \setminus x \cap y| = a_0 - a_1 \neq b_0 - b_1 = |h(x) \cap h(z) \setminus h(x) \cap h(y)|$. □

Corollary 12. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ, with $cf(\kappa) > \omega$, and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. If $\{|x \cap y| : \{x,y\} \in [\mathcal{A}]^2\}$ and $\{|x \cap y| : \{x,y\} \in [\mathcal{B}]^2\}$ are almost oscillating, there is no homeomorphism from $\Psi(\mathcal{A})$ to $\Psi(\mathcal{B})$ that extends h.*

Proof. If $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ is such homeomorphism, by Corollary 11 there is $\mathcal{A}' \in [\mathcal{A}]^\kappa$ such that $H \restriction \mathcal{A}' : \mathcal{A}' \to H[\mathcal{A}']$ is of dense oscillation. If $W = \bigcup_{A \in \mathcal{A}'} A$, then $Z = \mathcal{A}' \cup W$ is a subspace of $\Psi(\mathcal{A})$ and $H \restriction Z$ is a homeomorphism contradicting Proposition 8. □

3. $\mathfrak{c}$ many types of Ψ-spaces

Next we construct $\mathfrak{c}$ many AD families of the same size whose Ψ-spaces are pairwise non-homeomorphic for each of the classes of Luzin families and branch families of 2^ω.

Theorem 13. *There are $\mathfrak{c}$ different Luzin families (of size ω_1) with non-homeomorphic Ψ-spaces.*

Proof. Given $L = \{k_n : n \in \omega\} \subseteq \omega$ such that $k_n > \sum_{i<n} k_i$, construct a Luzin family $\mathcal{A}_L$ as follows: Choose a partition $\{A_n : n \in \omega\}$ of ω into infinite sets. Suppose constructed A_β, $\beta < \alpha$, with α an infinite countable ordinal. Let $\{B_n : n \in \omega\}$ be an enumeration with no repetitions of $\{A_\beta : \beta < \alpha\}$ and for each $n \in \omega$, pick $a_n \subseteq B_n \setminus \bigcup_{i<n} B_i$ such that $|(\bigcup_{i \leq n} a_i) \cap B_n| = k_n$. Let $A_\alpha = \bigcup_{n \in \omega} a_n$ and $\mathcal{A}_L = \{A_\alpha : \omega < \alpha < \omega_1\}$. It is easy to see that $\mathcal{A}_L$ is a Luzin family. Observe that

$$\forall \omega < \alpha, \beta < \omega_1 \; \exists n \in \omega \; (|A_\alpha \cap A_\beta| = k_n). \tag{2}$$

This is how we construct a Luzin family $\mathcal{A}_L$ from a given set of natural numbers L. All the Luzin families considered in the next are constructed from a fixed partition $\{A_n : n \in \omega\}$ of ω.

By Proposition 10, let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a pairwise almost oscillating family of sets of ω. For every $\alpha < \mathfrak{c}$, let $Q_\alpha = \{q_n^\alpha : n \in \omega\} \subseteq P_\alpha$ such that for every $n \in \omega$, $q_n^\alpha > \sum_{i<n} q_i^\alpha$. Notice that $\{Q_\alpha : \alpha < \mathfrak{c}\}$ is also a pairwise almost oscillating family of sets of ω. It follows from (2) that for any $\alpha < \mathfrak{c}$, $\{|x \cap y| : \{x,y\} \in [\mathcal{A}_{Q_\alpha}]^2\} \subseteq Q_\alpha$. Since "almost oscillating" is a hereditary property, for any $\omega < \beta, \alpha < \mathfrak{c}$, the sets $\{|x \cap y| : \{x,y\} \in [\mathcal{A}_{Q_\alpha}]^2\}$, $\{|x \cap y| : \{x,y\} \in [\mathcal{A}_{Q_\beta}]^2\}$ are almost oscillating. By Corollary 12, $\{\mathcal{A}_{Q_\alpha} : \alpha < \mathfrak{c}\}$ is the desired collection of Luzin families. □

Theorem 14. *Given a cardinal $\kappa \leq \mathfrak{c}$ of uncountable cofinality, there are $\mathfrak{c}$ different homeomorphic subsets of 2^ω of size κ with non-homeomorphic Ψ-spaces.*

Proof. Given $A \in [\omega]^\omega$, consider the tree $S_A \subseteq 2^{<\omega}$ defined by $\emptyset \in S_A$ and

$$s \in Lev_n(S_A) \implies (s^\frown 1 \in S_A) \wedge (s^\frown 0 \in S_A \iff n \in A).$$

Let X be any subset of size κ of the set of branches $[S_A] \subseteq 2^\omega$. Notice that

$$\forall x, y \in X \ (\Delta(x, y) = |\widehat{x} \cap \widehat{y}| \in A). \tag{3}$$

Again, by Proposition 10, let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a pairwise almost oscillating family of sets of ω. Note that if $A, B \in [\omega]^\omega$, then $[S_A] \simeq [S_B] \simeq 2^\omega$, and $A \cap B =^* \emptyset$ implies that $|[S_A] \cap [S_B]| < \omega$. Hence, we can choose $X_\alpha \in [[S_{P_\alpha}]]^\kappa$ such that the X_α's are all different, but $X_\alpha \simeq X_\beta$, whenever $\alpha, \beta < \mathfrak{c}$. It follows from (3) that $\{|\widehat{x} \cap \widehat{y}| : \{x, y\} \in [X_\alpha]^2\} \subseteq P_\alpha$, for $\alpha < \mathfrak{c}$ and so, the sets $\{|\widehat{x} \cap \widehat{y}| : \{x, y\} \in [X_\alpha]^2\}$, $\{|\widehat{x} \cap \widehat{y}| : \{x, y\} \in [X_\beta]^2\}$ are almost oscillating, for $\beta, \alpha < \mathfrak{c}$. By Corollary 12, $\{X_\alpha : \alpha < \mathfrak{c}\}$ is the desired collection of subsets of 2^ω. □

Corollary 15. *Let $\mathcal{A}$ be an AD family of size κ. If there are $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{A}]^\kappa$ such that $\{|x \cap y| : x, y \in \mathcal{A}_0\}$ and $\{|x \cap y| : x, y \in \mathcal{A}_1\}$ are almost oscillating, then $\Psi(\mathcal{A}) \not\simeq \Psi(\mathcal{A}_0)$.*

Proof. If $h : \mathcal{A}_0 \to \mathcal{A}$ is a bijection, use Corollary 11 to find $\mathcal{A}_0' \in [h^{-1}[\mathcal{A}_1]]^\kappa$ such that $h \restriction_{\mathcal{A}_0'} : \mathcal{A}_0' \to h[\mathcal{A}_0']$ is of dense oscillation. Now, it follows from Proposition 8 that there can not be a homeomorphism between $\Psi(\mathcal{A}_0')$ and $\Psi(h[\mathcal{A}_0'])$ that extends $h \restriction_{\mathcal{A}_0'}$. This implies that it can not be a homeomorphism between $\Psi(\mathcal{A}_0)$ and $\Psi(\mathcal{A})$ which extends h. □

Motivated by Corollary 15, we ask the following. A positive answer to it gives raise a negative answer to Question 2. However, we do not even know if CH answers:

Question 16. *Let $\mathcal{A}$ be an AD family on ω of size ω_1. Are there $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{A}]^{\omega_1}$ such that $\{|x \cap y| : \{x, y\} \in [\mathcal{A}_0]^2\}$ and $\{|x \cap y| : \{x, y\} \in [\mathcal{A}_1]^2\}$ are almost oscillating?*

The arguments under CH below Question 2 say that if $\mathcal{A}$ is an AD family of size ω_1, then there is $\mathcal{A}_0 \in [\mathcal{A}]^{\omega_1}$ such that $\Psi(\mathcal{A}) \not\simeq \Psi(\mathcal{A}_0)$. However, the sets $\{|x \cap y| : \{x, y\} \in [\mathcal{A}_0]^2\}$ and $\{|x \cap y| : \{x, y\} \in [\mathcal{A}]^2\}$ are far from being almost oscillating (the first is contained in the second).

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