

MARTIN'S AXIOM AND ω -RESOLVABILITY OF BAIRE SPACES

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August 12, 2009

ABSTRACT. We prove that, assuming MA , every crowded T_0 space X is ω -resolvable if it satisfies one of the following properties: (1) it contains a π -network of cardinality $< \mathfrak{c}$ constituted by infinite sets, (2) $\chi(X) < \mathfrak{c}$, (3) X is a T_2 Baire space and $c(X) \leq \aleph_0$ and (4) X is a T_1 Baire space and has a network \mathcal{N} with cardinality $< \mathfrak{c}$ and such that the collection of the finite elements in it constitutes a σ -locally finite family.

Furthermore, we prove that the existence of a T_1 Baire irresolvable space is equivalent to the existence of a T_1 Baire ω -irresolvable space, and each of these statements is equivalent to the existence of a T_1 almost- ω -irresolvable space.

Finally, we prove that the minimum cardinality of a π -network with infinite elements of a space $Seq(u_t)$ is strictly greater than \aleph_0 .

1. INTRODUCTION

Every space in this article is T_0 and crowded (that is, without isolated points) and so it is infinite. A space X is *resolvable* if it contains two dense disjoint subsets. A space which is not resolvable is called *irresolvable*. Resolvable and irresolvable spaces were studied extensively first by Hewitt [14]. Later, El'kin and Malyhin published a number of papers on these subjects and their connections with various topological problems. One of the problems considered by Malyhin in [21] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. He proved that there is such a space if and only if there is a space X on which every real-valued function is continuous at some point. (The question about the existence of a --Hausdorff-- space on which every real-valued function is continuous at some point was posed by M. Katětov in [16].) He also proved:

1. if we assume $V = L$, there is not a Baire irresolvable space,
2. if there is a measurable cardinal, then there is a Baire irresolvable space.

Bolstein introduced in [5] the spaces X in which it is possible to define a real-valued function f with countable range and such that f is discontinuous at every point of X (he called these spaces *almost resolvable*), and proved that every resolvable space satisfies this condition. It was proved in [12] that X is almost resolvable iff there is a function $f : X \rightarrow \mathbb{R}$ such that f is discontinuous at every point of X . *Almost- ω -resolvable* spaces were introduced in [25]; these are spaces in which it is possible to define a real-valued function f with countable range, and such that $r \circ f$ is discontinuous in every point of

1991 *Mathematics Subject Classification*. Primary: 54E52, 54A35. Secondary: 54D10, 54A10.

Key words and phrases. Martin's Axiom, Baire spaces, Resolvable spaces, ω -resolvable spaces, almost resolvable spaces, almost- ω -resolvable spaces, infinite π -network.

X , for every real-valued finite-to-one function r . It was proved in that article that for a Tychonoff space X , the space of real continuous functions with the box topology, $C_{\square}(X)$, is discrete if and only if X is almost- ω -resolvable. It was also proved that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost- ω -resolvable, and that, on the contrary, if $V = L$ then every crowded space is almost- ω -resolvable. Later, it was pointed out in [2], Corollary 5.4, that a Baire space is resolvable if and only if it is almost resolvable; so,

1.1. Theorem. *A Baire almost- ω -resolvable space is resolvable.*

It is unknown if every Baire almost- ω -resolvable space is 3-resolvable. With respect to this problem we have the following theorems.

1.2. Theorem. [23] *Gödel's axiom of constructibility, $V = L$, implies that every Baire space is ω -resolvable.*

1.3. Theorem. [2] *Every T_1 Baire space such that each of its dense subsets is almost- ω -resolvable is ω -resolvable.*

These last two results transform our problem to that of finding subclasses of Baire spaces such that each of its crowded dense subsets is almost- ω -resolvable, assuming axioms consistent with ZFC which contrast with $V = L$. Of course, a classic axiom with these characteristics is $MA + \neg CH$. This bet is strengthened by the following result due to V. I. Malykhin ([22, Theorem 1.2]):

1.4. Theorem. [$MA_{\text{countable}}$] *Let a topology on a countable set X have a π -network of cardinality less than \mathfrak{c} consisting of infinite subsets. Then this topology is ω -resolvable.*

It was proved in [2] that every space with countable tightness, every space with π -weight $\leq \aleph_1$ and every σ -space are hereditarily almost- ω -resolvable. So, by Theorem 1.3, every T_1 Baire space with either countable tightness or π -weight $\leq \aleph_1$ or σ is ω -resolvable.

In this article we are going to continue the study of almost- ω -resolvable and Baire resolvable spaces, and we will solve some problems related to these posed in [2]. Section 2 is devoted to establishing basic definitions and results. In Section 3 we prove that under MA every space with either π -weight $< \mathfrak{c}$ or $\chi(X) < \mathfrak{c}$ is ω -resolvable. Furthermore, we are going to see in Section 4 that under SH every T_2 Baire space with countable cellularity is ω -resolvable. Next, in Section 5 is devoted to analyse almost- ω -irresolvable spaces. We prove in this section that there is a T_1 Baire irresolvable space iff there is a T_1 Baire ω -irresolvable space, iff there is a T_1 almost- ω -irresolvable space. Finally in Section 6, we prove that the minimum cardinality of a π -network with infinite elements of a space $Seq(u_t)$ is strictly greater than \aleph_0 . Moreover, we propose several problems related to our matter through the article.

2. BASIC DEFINITIONS AND PRELIMINARIES

A space X is *resolvable* if it is the union of two disjoint dense subsets. We say that X is *irresolvable* if it is not resolvable. For a cardinal number $\kappa > 1$, we say that X is *κ -resolvable* if X is the union of κ pairwise disjoint dense subsets.

The *dispersion character* $\Delta(X)$ of a space X is the minimum of the cardinalities of nonempty open subsets of X . If X is $\Delta(X)$ -resolvable, then we say that X is *maximally resolvable*. A space X is *hereditarily irresolvable* if every subspace of X is irresolvable. And X is *open-hereditarily irresolvable* if every open subspace of X is irresolvable.

We call a space (X, t) *maximal* if (X, t') contains at least one isolated point when t' strictly contains the topology t . And a space X is *submaximal* if every dense subset of X is open. Moreover, maximal spaces are submaximal, and these are hereditarily irresolvable spaces, which in turn are open-hereditarily irresolvable.

It is possible to prove that a space X is *almost resolvable* if and only if X is the union of a countable collection of subsets each of them with an empty interior [5].

It was proved in [25] that the following formulation can be given as a definition of almost- ω -resolvable space: A space X is called *almost- ω -resolvable* if X is the union of a countable collection $\{X_n : n < \omega\}$ of subsets in such a way that for each $m < \omega$, $\text{int}(\bigcup_{i \leq m} X_i) = \emptyset$. In particular, every almost- ω -resolvable space is almost resolvable, every ω -resolvable space is almost- ω -resolvable, every almost resolvable space is infinite, and every T_1 separable space is almost- ω -resolvable.

We are going to say that a space X is *hereditarily almost- ω -resolvable* if each crowded subspace of X is almost- ω -resolvable, and X is *dense-hereditarily almost- ω -resolvable* if each crowded dense subspace of X is almost- ω -resolvable.

Let X be a κ -resolvable (resp., almost-resolvable, almost- ω -resolvable) space. A κ -*resolution* (resp., an *almost resolution*, an *almost- ω -resolution*) for X is a partition $\{V_\alpha : \alpha < \kappa\}$ (resp., a partition $\{V_n : n < \omega\}$) of X such that each V_α is a dense subset of X (resp., $\text{int}(V_n) = \emptyset$ for every $n < \omega$, $\text{int}(\bigcup_{i=0}^n V_i) = \emptyset$ for every $n < \omega$).

Finally, a space X is *almost- ω -irresolvable* (resp., κ -irresolvable) if X is not almost- ω -resolvable (resp., X is not κ -resolvable). The hereditary version of almost- ω -irresolvability or κ -irresolvability is that which states that every crowded subspace of X is not almost- ω -resolvable, and, respectively, is not κ -resolvable.

2.1. Example. There are non- T_0 topological spaces which are almost resolvable but not almost- ω -resolvable. In fact, let X be an infinite set and $x, y \in X$ with $x \neq y$. We define a collection \mathcal{T} of subsets of X as follows: $A \in \mathcal{T}$ if either A is the empty set or $x, y \in A$. The family \mathcal{T} is a topology in X and (X, \mathcal{T}) satisfies the required conditions.

2.2. Example. It was proved in Theorem 4.4 of [18] that the existence of an ω_1 -complete ideal \mathcal{I} over ω_1 which has a dense set of size ω_1 implies the existence of a T_2 Baire strongly irresolvable topology \mathcal{T} on ω_1 . On the other hand, it was observed in [25, Corollary 4.9] that every Baire irresolvable space is not almost resolvable. Therefore, (ω_1, \mathcal{T}) is not almost resolvable.

2.3. Example. If there is a measurable cardinal κ , then there is a resolvable Baire space X which is not almost- ω -resolvable and $\Delta(X) = \kappa$. Indeed, let κ be a non-countable Ulam-measurable cardinal, and let p be a free ultrafilter on κ ω_1 -complete. Let $X = \kappa \cup \{p\}$. We define a topology t for X as follows: $A \in t \setminus \{\emptyset\}$ if and only if $p \in A$ and $A \cap \kappa \in p$. This space is a Baire resolvable non-almost- ω -resolvable space with $\Delta(X) = \alpha$. Now, let \mathcal{T} be equal to $\{A \subseteq X : A \cap \kappa \in p\}$; \mathcal{T} is a topology in X too, and (X, \mathcal{T}) is T_1 submaximal, Baire with $\Delta(X) = \alpha$, but it is not almost resolvable.

Related to the last examples we have:

2.4. Question. *Is there a T_2 resolvable Baire space which is not almost- ω -resolvable?*

2.5. Examples. In **ZFC**, there are almost- ω -resolvable spaces which are not resolvable. Indeed, the union of Tychonoff crowded topologies in \mathbb{Q} generates a Tychonoff crowded topology. By Zorn's Lemma, we can consider a maximal Tychonoff topology \mathcal{T} in \mathbb{Q} . The space $(\mathbb{Q}, \mathcal{T})$ is countable (so, almost- ω -resolvable) hereditarily irresolvable spaces ([14, Theorems 15 and 26], [8, Example 3.3]). $(\mathbb{Q}, \mathcal{T})$ is Tychonoff.

In [1], the authors construct by transfinite recursion a “concrete” (in the sense that we can say how its open sets look) example of a countable dense subset X of the space $2^{\mathfrak{c}}$ which is irresolvable. Since X is countable, it is almost- ω -resolvable.

2.6. Example. For every cardinal number κ , there exists a Tychonoff space X which is almost- ω -resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$. In fact, let λ be a cardinal number such that $\kappa \leq \lambda$ and $\text{cof}(\lambda) = \aleph_0$. Let H, G and τ be the topological groups and the topology in G , respectively, defined in [11] pages 33 and 34, with $|H| = \lambda$. L. Feng proved there that $(H, \tau|_H)$ is an irresolvable card-homogeneous (every open subset of H has the same cardinality as H) Tychonoff space, and each subset $S \subseteq H$ with cardinality strictly less than λ is a nowhere dense subset of H . Let $(\lambda_n)_{n < \omega}$ be a sequence of cardinal numbers such that $\lambda_n < \lambda_{n+1}$ for every $n < \omega$ and $\sup\{\lambda_n : n < \omega\} = \lambda$. We take subsets H_n of H with the properties $H_n \subseteq H_{n+1}$ and $|H_n| = \lambda_n$ for each $n < \omega$, and $H = \bigcup_{n < \omega} H_n$. We have that each H_n is nowhere dense in H ; so $\{H_n : n < \omega\}$ is an almost- ω -resolvable sequence on H . That is, H is almost- ω -resolvable. By the Hewitt Decomposition Theorem (see [14, Theorem 28]), there exists a nonempty open subset U of H which is hereditarily irresolvable. Besides, $\Delta(U) = \Delta(H) \geq \kappa$ and U is almost- ω -resolvable.

2.7. Examples. The first example of a Hausdorff maximal group was constructed by Malykhin in [20] under Martin’s Axiom. Malykhin also constructed in [22], in the BK model M_{ω_1} (see [3]) a topological group topology \mathcal{T}' in the infinite countable Boolean group Ω of all finite subsets of ω with symmetric difference as the group operation, such that (Ω, \mathcal{T}') is T_2 , irresolvable and its weight is ω_1 (compare with Corollary 3.6 below). Moreover, in M_{ω_1} , $\omega_1 < \mathfrak{c}$. Moreover, he constructed in M_{ω_1} a countable irresolvable dense subset in 2^{ω_1} . This space has of course weight ω_1 .

On the other hand, the class of resolvable spaces includes spaces with well known properties:

2.8. Theorem. (1) If X has a π -network \mathcal{N} such that $|\mathcal{N}| \leq \Delta(X)$ and each $N \in \mathcal{N}$ satisfies $|N| \geq \Delta(X)$, then X is maximally resolvable [9].

(2) Hausdorff k -spaces are maximally resolvable [24].

(3) Countably compact regular T_1 spaces are ω -resolvable [7].

(4) Arc connected spaces are ω -resolvable.

(5) Every biradial space is maximally resolvable [28].

(6) Every homogeneous space containing a non trivial convergent sequence is ω -resolvable [27].

(7) If G is an uncountable \aleph_0 -bounded topological group, then G is \aleph_1 -resolvable [28].

(8) T_1 Baire spaces with either countable tightness or π -weight $\leq \aleph_1$ are ω -resolvable [2].

The following basic results will be very helpful (see, for example, [6]).

2.9. Propositions. (1) If X is the union of κ -resolvable (resp., almost-resolvable, almost- ω -resolvable) subspaces, then X has the same property.

(2) Every open and every regular closed subset of a κ -resolvable (resp., almost resolvable, almost- ω -resolvable) space shares this property.

(3) Let X be a space which contains a dense subset which is κ -resolvable (resp., almost resolvable, almost- ω -resolvable). Then, X satisfies this property too.

The following results are easy to prove and are well known.

2.10. Proposition. *Let Y be a κ -resolvable (resp., almost-resolvable, almost- ω -resolvable) space. If $f : X \rightarrow Y$ is a continuous and onto function, and for each open subset A of X the interior of $f[A]$ is not empty, then X is κ -resolvable (resp., almost-resolvable, almost- ω -resolvable).*

2.11. Proposition. *Let $f : X \rightarrow Y$ be continuous and bijective. If X is κ -resolvable (resp., almost-resolvable, almost- ω -resolvable), so is Y .*

2.12. Proposition. (1) *If X is κ -resolvable (resp., almost resolvable, almost- ω -resolvable) and Y is an arbitrary topological space, then $X \times Y$ is κ -resolvable (resp., almost resolvable, almost- ω -resolvable).*

(2) [2] *If X and Y are almost resolvable, then $X \times Y$ is resolvable.*

(3) (O. Masaveu) *If X is the product space $\prod_{\alpha < \kappa} X_\alpha$ where $\kappa \geq \omega$ and each X_α has more than one point, then X is 2^κ -resolvable.*

The following lemmas will be useful later.

2.13. Proposition. *If X is a crowded space such that $\text{cof}(|X|) = \aleph_0$ and every open subset of X has cardinality $|X|$, then X is almost- ω -resolvable.*

2.14. Proposition. *If X has tightness equal to κ , then each point $x \in X$ is contained in a crowded subset of X of cardinality $\leq \kappa$.*

Proof. Let $x_0 \in X$ be an arbitrary fixed point. Since X is crowded, $x_0 \in \text{cl}_X[X \setminus \{x_0\}]$; so there is a subset $F_1 \subseteq X \setminus \{x_0\}$ of cardinality $\leq \kappa$ such that $x_0 \in \text{cl}_X F_1$. If $F_0 \cup F_1$ is crowded, where $F_0 = \{x_0\}$, then we have finished. Otherwise, for each isolated point x of $F_0 \cup F_1$, there is a subset $F_x^2 \subseteq X \setminus (\{x_0\} \cup F_1)$ of cardinality $\leq \kappa$ such that $x \in \text{cl}_X F_x^2$. Let $F_2 = \bigcup_{x \in G_1} F_x^2$ where G_1 is the set of isolated points of $F_0 \cup F_1$. Again, there are two possible situations: either $F_0 \cup F_1 \cup F_2$ is a crowded subspace of cardinality $\leq \kappa$ containing x_0 , or $G_2 = \{x \in F_2 : x \text{ is an isolated point of } F_0 \cup F_1 \cup F_2\}$ is not empty. In this last case, for each $x \in G_2$ we take a subset $F_x^3 \subseteq X \setminus (F_0 \cup F_1 \cup F_2)$ of cardinality $\leq \kappa$ for which $x \in \text{cl}_X F_x^3$. We write $F_3 = \bigcup_{x \in G_2} F_x^3$. Continuing this process if necessary, we obtain either a finite sequence F_0, \dots, F_n of subsets of X such that $x_0 \in F = \bigcup_{0 \leq i \leq n} F_i$ and F has cardinality $\leq \kappa$ and crowded, or we have to go further: $x_0 \in F = \bigcup_{n < \omega} F_n$. In this last case too, F has cardinality $\leq \kappa$ and is crowded. \square

3. MARTIN'S AXIOM, π -NETWEIGHT AND ω -RESOLVABLE SPACES

First, in this section we are going to present, by using Martin's Axiom, a generalization of Theorem 1.4. As usual, if I and J are two sets, $F_n(I, J)$ stands for the collection of the finite functions with domain contained in I and range contained in J . We define a partial order \leq in $F_n(I, J)$ by letting $p \leq q$ iff $p \supseteq q$. The partial order set $(F_n(I, J), \leq)$ is ccc if and only if $|J| \leq \aleph_0$ (Lemma 5.4, p. 205 in [17]).

Let (X, τ) be a topological space. A collection $\mathcal{N} \subseteq \mathcal{P}(X)$ is a π -network of X if each element $U \in \tau \setminus \{\emptyset\}$ contains an element of \mathcal{N} .

3.1. Definitions. Let κ be an infinite cardinal.

- (1) A space X is almost- κ -resolvable if X can be partitioned as $X = \bigcup_{\alpha < \kappa'} X_\alpha$ where $\omega \leq \kappa' \leq \kappa$, $X_\alpha \neq \emptyset$, and $X_\alpha \cap X_\xi = \emptyset$ if $\alpha \neq \xi$, such that every non-empty open subset of X has a non empty intersection with an infinite collection of elements in $\{X_\alpha : \alpha < \kappa\}$.
- (2) Let $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$ be a partition of X . A collection $\mathcal{N} = \{N_\xi : \xi < \tau\}$ of infinite subsets of κ is a π -network of \mathcal{X} if for each open set U of X , $\{\alpha < \kappa : X_\alpha \cap U \neq \emptyset\} \supseteq N_\xi$ for a $\xi < \tau$.
- (3) A space X is called precisely almost- κ -resolvable if X contains a resolution with a π -network \mathcal{N} such that $|\mathcal{N}| \leq \kappa$.

The following well known result is due to K. Kuratowski.

3.2. Lemma. (The disjoint refinement lemma) Let $\{A_\xi : \xi < \kappa\}$ be a collection of sets such that for each $\xi < \kappa$ $|A_\xi| \geq \kappa$. Then, there is a collection $\{B_\xi : \xi < \kappa\}$ of sets satisfying:

- (1) $B_\xi \subseteq A_\xi$ for all $\xi < \kappa$,
- (2) $|B_\xi| = \kappa$ for all $\xi < \kappa$,
- (3) $B_\xi \cap B_\zeta = \emptyset$ for $\xi, \zeta < \kappa$ with $\xi \neq \zeta$.

3.3. Proposition. A space X is precisely almost- ω -resolvable if and only if X is ω -resolvable.

Proof. Let X be a precisely almost- ω -resolvable space. Let $\mathcal{X} = \{X_\xi : \xi < \tau\}$ be a precise partition of X , and $\mathcal{M} = \{M_n : n < \omega\}$ be a π -network of \mathcal{X} . Because of Lemma 3.2, there are infinite and pairwise disjoint sets $T_0, T_1, \dots, T_n, \dots$ such that $T_i \subseteq M_i$ for all $i < \omega$.

For each $n < \omega$, we faithfully enumerate T_n : $\{k_i^n : i < \omega\}$. Now we define for each $i < \omega$, $D_i = \bigcup_{j < \omega} X_{k_i^j}$. Each D_n is dense in X and $D_i \cap D_j = \emptyset$ if $i \neq j$.

Moreover, if X is ω -resolvable and $\mathcal{D} = \{D_n : n < \omega\}$ is a collection of pairwise disjoint dense subsets of X , then \mathcal{D} is a precise partition of X and $\mathcal{M} = \{\omega\}$ is a π -network of \mathcal{D} . \square

When we assume Martin's Axiom, we can generalize Proposition 3.3:

3.4. Theorem. Let $\mathcal{X} = \{X_\alpha : \alpha < \tau\}$ be an almost- τ -resolvable partition of X . Let $\mathcal{N} = \{N_\xi : \xi < \kappa\}$ be a π -network of \mathcal{X} such that $\kappa < \mathfrak{c}$. If we assume Martin's Axiom, then X is ω -resolvable. In particular, MA implies that ω -resolvability and almost- κ -resolvability precise coincide when $\kappa < \mathfrak{c}$.

Proof. In this case, we put $\mathbb{P} = (Fn(\kappa, \omega), \leq)$ where \leq is defined at the beginning of this section. For each $k \in \omega$ and $N \in \mathcal{N}$, we take the set

$$D_N^k = \{p \in \mathbb{P} : \exists \xi \in N \text{ such that } p(\xi) = k\}.$$

It happens that each D_N^k is dense in \mathbb{P} . In fact, let q be an arbitrary element in \mathbb{P} . We can take $\xi \in N \setminus \text{dom}(q)$ because N is infinite. The function $p = q \cup \{(\xi, k)\}$ belongs to D_N^k and is less than q .

The partial ordered set \mathbb{P} is ccc and $\mathcal{D} = \{D_N^k : k < \omega, N \in \mathcal{N}\}$ has cardinality strictly less than \mathfrak{c} . So, there exists a \mathcal{D} -generic filter G in \mathbb{P} . Take $f = \bigcup G$. Then $f : \kappa \rightarrow \omega$ is onto and $\kappa = \bigcup_{n < \omega} Y_n$ where $Y_n = f^{-1}[\{n\}]$.

Now, for each $n < \omega$, we consider the set $X_n = \bigcup_{\alpha \in Y_n} X_\alpha$. It is easy to prove that $\{X_n : n < \omega\}$ is a partition of $\bigcup_{n < \omega} X_n$. Moreover, each X_n is a dense subset of X . Indeed,

let n_0 be a natural number. We are going to prove that X_{n_0} is dense. Let U be an open set of X . Because of the properties of \mathcal{N} , there is $N_0 \in \mathcal{N}$ such that $\{\alpha < \tau : X_\alpha \cap U \neq \emptyset\} \supseteq N_0$. We take $p \in D_{N_0}^{n_0} \cap G$. It happens that there is a $\xi \in N_0$ such that $p(\xi) = n_0$. Hence, $f(\xi) = n_0$. This means that $\xi \in f^{-1}[\{n_0\}] = Y_{n_0}$. By definition, X_ξ must have a non-empty intersection with U , and therefore $U \cap X_{n_0} = U \cap \bigcup_{\alpha \in Y_{n_0}} X_\alpha \neq \emptyset$. \square

Assume that $\{x_\xi : \xi < \tau\}$ is a faithful enumeration of a space X . If X possesses a π -network \mathcal{N} with infinite elements, the collection $\{M_N : N \in \mathcal{N}\}$ where $M_N = \{\xi < \tau : x_\xi \in N\}$, is a π -network of the partition $\{\{x_\xi\} : \xi < \tau\}$. So the following result is a corollary of Theorem 3.4.

3.5. Theorem. *Let X be a crowded topological space with a π -network \mathcal{N} with cardinality $\kappa < \mathfrak{c}$ and such that each element in \mathcal{N} is infinite. If we assume Martin's Axiom, then X is an ω -resolvable space.*

Recall that every biradial space is maximally resolvable. Moreover, every space with $\pi w(X) \leq \Delta(X)$ is maximally resolvable (see [4]). With respect to these ideas we have:

3.6. Corollary. *[MA] Every crowded space X with π -weight $< \mathfrak{c}$ is ω -resolvable. In particular, every space with weight $< \mathfrak{c}$ is hereditarily ω -resolvable.*

Proof. Let \mathcal{N} be a π -base of X of cardinality $< \mathfrak{c}$. Since X is crowded and each element of \mathcal{N} is open in X , then $|N| \geq \aleph_0$ for each $N \in \mathcal{N}$. On the other hand, \mathcal{N} is a π -network in X , so the conclusion follows. \square

It is easy to see that if X has π -character and density $\leq \kappa$, then X has a π -base of cardinality $\leq \kappa$.

3.7. Proposition. *[MA] If X is a space with density and π -character $< \mathfrak{c}$, then every dense subset of X is ω -resolvable.*

Proof. The space X has a π -base \mathcal{B} of cardinality $< \mathfrak{c}$. Let H be an arbitrary dense subset of X . It happens now that $\mathcal{M} = \{N \cap H : N \in \mathcal{B}\}$ is a π -base of H and has cardinality $< \mathfrak{c}$. So, by Corollary 3.6, H is ω -resolvable. \square

For every space X , $\max\{t(X), \pi\chi(X)\} \leq \chi(X)$, so, as a consequence of the last result, and related to Theorems 2.8.2 and 2.8.8, we have:

3.8. Theorem. *[MA] If X is a space such that $\chi(x, X) < \mathfrak{c}$ for each $x \in X$, then X is hereditarily ω -resolvable.*

Proof. Let Y be a crowded subspace of X . The character of Y is strictly less than \mathfrak{c} ; thus, the tightness of Y is $< \mathfrak{c}$. Hence, each point y in Y is contained in a crowded subspace Y_y of Y of cardinality $< \mathfrak{c}$ (Proposition 2.14). The density and character of each Y_y is strictly less than \mathfrak{c} . By Proposition 3.7, Y_y is ω -resolvable. Then Y is ω -resolvable (see Proposition 2.9.1). \square

We can go a little further than Theorem 3.8. For each point $x \in X$ we define $R(x, X) = \min\{|\Lambda| : \Lambda \text{ is a directed partially ordered set and there is a net } (x_\alpha)_{\alpha \in \Lambda} \text{ in } X \setminus \{x\} \text{ such that } (x_\alpha)_{\alpha \in \Lambda} \text{ converges to } x \text{ in } X\}$. And we define $R(X) = \sup_{x \in X} R(x, X)$. Of course, for each $x \in X$, $R(x, X) \leq \chi(x, X)$. Since MA implies that \mathfrak{c} is a regular cardinal, we have that, by Theorem 3.5, MA implies that every space X containing a dense subset Y of cardinality $\leq \kappa < \mathfrak{c}$ and such that for every $y \in Y$, $R(y, X) < \mathfrak{c}$, is ω -resolvable. This result can be ameliorated. Indeed, by using a similar proof to that of Proposition 2.14, if X is a space with $R(x, X) \leq \kappa$ for each $x \in X$, then each point $x \in X$ is contained in a crowded subspace X_x of X of cardinality $\leq \kappa$ and having, for each $y \in X_x$, $R(y, X_x) \leq \kappa$. So:

3.9. Corollary. [MA] *Let X be a space such that for every $x \in X$, $R(x, X) < \mathfrak{c}$, then X is ω -resolvable.*

In Proposition 4.5 of [2] was proved that every T_2 σ -space is almost- ω -resolvable. When X has a countable network, we can repeat that proof assuming only the weaker condition T_0 . So every space with countable network is almost- ω -resolvable. With respect to σ -spaces, Proposition 4.5 in [2] and Martin's Axiom, Theorem 3.9 allows us to say something else which is, in some sense, stronger than Theorem 3.5:

3.10. Proposition. [MA] *Let κ be an infinite cardinal $< \mathfrak{c}$. Let X be a space with a network \mathcal{N} such that for each finite subcollection \mathcal{N}' of \mathcal{N} , $\bigcap \mathcal{N}'$ is infinite or empty, and for each $x \in X$, $|\{N \in \mathcal{N} : x \in N\}| \leq \kappa$. Then, X is hereditarily ω -resolvable.*

Proof. Space X is the condensation of a crowded space Y (Y is X with the topology generated by \mathcal{N} as a base) which has character strictly less than \mathfrak{c} (see Proposition 2.11). \square

Next, we obtain a result that we can locate between Theorem 3.5 which deals with π -networks and Corollary 3.6 which speaks of bases. First a definition and some remarks. A space X is called σ -locally finite if X can be written as $\bigcup_{n < \omega} X_n$ where, for each $n < \omega$, the collection $\{\{x\} : x \in X_n\}$ is locally finite in X . It can be proved that a σ -locally finite crowded space is hereditarily almost- ω -resolvable.

3.11. Theorem. [MA] *Let X be a crowded topological space with a network \mathcal{N} with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is σ -locally finite in $\bigcup \mathcal{N}_0$. Then X can be written as $Y_0 \cup Y_1$ where Y_0 is a (possibly empty) regular closed ω -resolvable subspace and Y_1 is an open (possibly empty) almost- ω -resolvable, hereditarily ω -irresolvable space. Besides, if Y_1 is not void, it contains a non-empty open subset which is hereditarily almost- ω -resolvable. Moreover, if X is a T_1 Baire space, then X must be ω -resolvable.*

Proof. Let \mathcal{M} be the collection of all subspaces of X which are ω -resolvable. Take $Y_0 = cl_X \bigcup \mathcal{M}$ and $Y_1 = X \setminus Y_0$. Of course Y_0 is closed and ω -resolvable. Now, if Y_1 is empty, we have already finished; if the contrary happen, Y_1 is hereditarily ω -irresolvable and the collection $\mathcal{N}' = \{N \in \mathcal{N} : N \subseteq Y_1\}$ is a network in Y_1 with cardinality $< \mathfrak{c}$ and such that $\mathcal{N}'_0 = \{N \in \mathcal{N}' : |N| < \aleph_0\}$ is σ -locally finite in $\bigcup \mathcal{N}'_0$. Of course \mathcal{N}'_0 is not empty, because otherwise, by Theorem 3.5, Y_1 would be ω -resolvable, but this is not possible. Let Z be the subspace $\bigcup_{N \in \mathcal{N}'_0} N$ of X . The space Z is σ -locally finite. Since Y_1 is hereditarily ω -irresolvable, Z is a dense subset of Y_1 . Then, Y_1 is almost- ω -resolvable. Furthermore, there must exist a non-empty open subset U of Y_1 such that each element of \mathcal{N}' contained in U is finite because otherwise Y_1 would be ω -resolvable (again by Theorem 3.5). So, $intZ$ is a non empty open subset which is hereditarily almost- ω -resolvable.

Assume now that X is T_1 and satisfies all the conditions of our proposition including the Baire property. In this case Y_1 must be empty because if this is not the case, the subspace $intZ$ of Y_1 would be a T_1 Baire hereditarily almost- ω -resolvable space. But this means, by Theorem 1.3, that $intZ$ is ω -resolvable, which is not possible. \square

If we consider in the previous theorem π -networks instead of networks, still we get something interesting.

3.12. Proposition. [MA] *Let X be a crowded topological space with a π -network \mathcal{N} with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is σ -locally finite. Then X is*

equal to $X_0 \cup X_1$ where $X_0 \cap X_1 = \emptyset$, X_0 is a regular closed (possibly empty) almost- ω -resolvable space and X_1 is an open (possibly empty) ω -resolvable subspace. In particular, X is, in this case, almost- ω -resolvable.

Proof. Let Y be the subspace $\bigcup_{N \in \mathcal{N}_0} N$. The space Y is σ -locally finite. If Y is empty, we obtain our result by Theorem 3.5. If Y is crowded, then it is almost- ω -resolvable (see Theorem 3.5 in [25]). If Y is not empty and is not crowded, we can find an ordinal number $\alpha > 0$ and, for each $\beta < \alpha$, an ω -resolvable subspace M_β of X such that $X_0 = cl_X(Y \cup cl_X(\bigcup_{\beta < \alpha} M_\beta))$ is almost- ω -resolvable. In fact, let D_0 be the set of isolated points in $Y_0 = Y$. For each $x \in D_0$, there is an open set A_x in X such that $A_x \cap Y_0 = \{x\}$. Observe that $A_x \setminus \{x\}$ is a dense subset of A_x and it satisfies the conditions in Theorem 3.5, so it is ω -resolvable. Thus, $M_0 = cl_X(\bigcup_{x \in D_0} A_x)$ is an ω -resolvable space. Assume that we have already constructed ω -resolvable subspaces M_β of X with $\beta < \gamma$. Put $Y_\gamma = Y \setminus cl_X(\bigcup_{\beta < \gamma} M_\beta)$. If Y_γ is empty or crowded, we take $\alpha = \gamma$, and in this case $cl_X(Y \cup cl_X(\bigcup_{\beta < \gamma} M_\beta))$ is almost- ω -resolvable because Y_γ is empty or crowded and σ -locally finite. If Y_γ is not empty and is not crowded, let D_γ be the set of isolated points in Y_γ . For each $x \in D_\gamma$ there is an open set A_x in X such that $A_x \cap Y_\gamma = \{x\}$ and $A_x \cap cl_X(\bigcup_{\beta < \gamma} M_\beta) = \emptyset$. Again $A_x \setminus \{x\}$ is a dense subset of A_x and it is ω -resolvable because of Theorem 3.5. Thus, $M_\gamma = cl_X(\bigcup_{x \in D_\gamma} A_x)$ is an ω -resolvable space. Continuing with this process we have to find an ordinal number α for which $X_0 = cl_X(Y \setminus cl_X(\bigcup_{\beta < \alpha} M_\beta))$ is almost- ω -resolvable.

Now, if $X_1 = X \setminus X_0$ is not empty, then it is a crowded space and $\mathcal{N}_1 = \{N \in \mathcal{N} : N \subseteq X_1\}$ is a π -network in X_1 with infinite elements and $|\mathcal{N}_1| < \mathfrak{c}$. Then, again by Theorem 3.5, X_1 is ω -resolvable. Therefore, $X = X_0 \cup X_1$, and X_0, X_1 satisfy the conditions of our proposition. \square

3.13. Questions.

- (1) Let X be a crowded space with cardinality $< \mathfrak{c}$. Does $MA + \neg CH$ imply that X is almost- ω -resolvable?
- (2) Is there a combinatorial axiom on ω_1 which assure us that every card-homogeneous topology in ω_1 is almost- ω -resolvable?
- (3) Does \diamond imply that every card-homogeneous topology in ω_1 is almost- ω -resolvable?

4. MARTIN'S AXIOM, CELLULARITY AND ω -RESOLVABLE BAIRE SPACES

It is well known that $MA(\omega_1)$ implies that a Souslin line does not exist. That is, $MA(\omega_1) \Rightarrow SH$. We show that it is enough to assume SH in order to prove that every T_2 space with countable cellularity is almost- ω -resolvable.

4.1. Theorem. *[SH] Every crowded T_2 space with countable cellularity is almost- ω -resolvable.*

Proof. Let $a_0 \in X$ and $F_0 = \{a_0\}$. Let \mathcal{C}_0 be a maximal cellular family of open sets in $X \setminus F_0$ containing at least two elements. Let X_0 be equal to $\bigcup \mathcal{C}_0$. Assume that we have already constructed, by recursion, families $\{\mathcal{C}_\alpha : \alpha < \gamma\}$, $\{X_\alpha : \alpha < \gamma\}$ and $\{F_\alpha : \alpha < \gamma\}$, such that

- (1) for all $\alpha < \gamma$, \mathcal{C}_α is a maximal cellular collection of open sets in X ,
- (2) if $\alpha < \xi < \gamma$, then \mathcal{C}_ξ properly refines \mathcal{C}_α .
- (3) if $\alpha < \xi < \gamma$ and $C \in \mathcal{C}_\alpha$, then \mathcal{C}_ξ contains a maximal cellular family of proper open sets of C having more than one element.
- (4) $X_\alpha = \bigcup \mathcal{C}_\alpha$ for each $\alpha < \gamma$,

- (5) the family $\{X_\alpha : \alpha < \gamma\}$ is a strictly decreasing γ -sequence of open sets in X ,
- (6) $F_\alpha \neq \emptyset$ for every $\alpha < \gamma$,
- (7) $F_\alpha \subseteq (\bigcap_{\xi < \alpha} X_\xi) \setminus X_\alpha$ for all $\alpha < \gamma$,
- (8) $\text{int}(F_\alpha) = \emptyset$ for all $\alpha < \gamma$.

If γ is a successor ordinal, say $\gamma = \xi + 1$, take for each $C \in \mathcal{C}_\xi$ a point $a_C^\gamma \in C$. Now, take a maximal cellular family of open proper subsets in $C \setminus \{a_C^\gamma\}$ with more than one element, \mathcal{C}_C^ξ (this is possible because C is T_2 and infinite). Put $\mathcal{C}_\gamma = \bigcup_{C \in \mathcal{C}_\xi} \mathcal{C}_C^\xi$, $X_\gamma = \bigcup \mathcal{C}_\gamma$ and $F_\gamma = \{a_C^\gamma : C \in \mathcal{C}_\xi\}$.

If γ is a limit ordinal, analyse the set $\bigcap_{\xi < \gamma} X_\xi$: if $\text{int}(\bigcap_{\xi < \gamma} X_\xi) = \emptyset$, declare our process finished; and if $\text{int}(\bigcap_{\xi < \gamma} X_\xi)$ is not empty, take a point $a_\gamma \in \text{int}(\bigcap_{\xi < \gamma} X_\xi)$ and take a maximal cellular family \mathcal{C}_γ with cardinality bigger than one of open proper subsets in $\text{int}(\bigcap_{\xi < \gamma} X_\xi) \setminus F_\gamma$ where $F_\gamma = \{a_\gamma\}$. Put $X_\gamma = \bigcup \mathcal{C}_\gamma$.

In this way we can find an ordinal number α_0 and families $\mathfrak{C} = \{\mathcal{C}_\alpha : \alpha < \alpha_0\}$, $\mathcal{X} = \{X_\alpha : \alpha < \alpha_0\}$ and $\mathcal{F} = \{F_\alpha : \alpha < \alpha_0\}$ satisfying properties from (1) to (8) above where α_0 is an ordinal number such that $\text{int}(\bigcap_{\xi < \alpha_0} X_\xi) = \emptyset$ and for each $\alpha < \alpha_0$, $\text{int}(\bigcap_{\xi < \alpha} X_\xi) \neq \emptyset$.

First, observe that α_0 must be a limit ordinal and every X_α is an open set of X . Now, consider the collection $\mathcal{Y} = \{Y_\alpha : \alpha < \alpha_0\}$ of subspaces of X where $Y_0 = X \setminus X_0$, and $Y_\alpha = (\bigcap_{\xi < \alpha} X_\xi) \setminus X_\alpha$ if $\alpha > 0$. We have that $F_\alpha \subseteq Y_\alpha$ and $\text{int}(Y_\alpha) = \emptyset$ for every $\alpha < \alpha_0$.

The set $\bigcup_{\alpha < \alpha_0} \mathcal{C}_\alpha$ with the order relation \subseteq is a tree T and each element in it has at least two immediate successors.

Claim 1: The height of T , α_0 , is at most $c(X)^+ = \omega_1$.

In fact, if $\alpha_0 > \omega_1$, then $\mathcal{C}_{\omega_1} \neq \emptyset$. Take $C_{\omega_1} \in \mathcal{C}_{\omega_1}$. Let $\mathcal{C} = \{C \in T : C \supseteq C_{\omega_1} \text{ and } C \neq C_{\omega_1}\}$. Since T is a tree, \mathcal{C} is a well ordered set with order type ω_1 . We can rename \mathcal{C} as $\{C_\alpha : \alpha < \omega_1\}$ where C_α is the only element in \mathcal{C}_α which belongs to \mathcal{C} . For each $\alpha < \omega_1$, there is $A_{\alpha+1} \in \mathcal{C}_{\alpha+1}$ such that $A_{\alpha+1} \subseteq C_\alpha$ and $A_{\alpha+1} \cap C_{\alpha+1} = \emptyset$. The set $\mathcal{A} = \{A_{\alpha+1} : \alpha < \omega_1\}$ is an antichain in T . Indeed, let $A_{\alpha+1}$ and $A_{\xi+1}$ be two different elements of \mathcal{A} . Assume that $\alpha < \xi$. Hence, $A_{\xi+1} \subseteq C_\xi$ and $C_\xi \subseteq C_{\alpha+1}$. But $C_{\alpha+1} \cap A_{\alpha+1} = \emptyset$. Therefore, $A_{\alpha+1} \cap A_{\xi+1} = \emptyset$. This means that $c(X) > \aleph_0$, which is a contradiction. We get that every chain and every antichain of T has cardinality $\leq \aleph_0$. Since we are assuming the Souslin's Hypothesis, there are no Souslin trees. Therefore $\alpha_0 < \omega_1$.

It is not difficult to prove that the set $Z = X \setminus X_{\alpha_0}$ is equal to $\bigcup_{\alpha < \alpha_0} Y_\alpha$ and that the collection $\{Y_\alpha : \alpha < \alpha_0\}$ is a partition of Z .

Claim 2: The collection $\{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\}$ is an almost- ω -resolution for X ; that is, X is almost- ω -resolvable.

The collection $\mathcal{Y} = \{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\}$ is a countable partition of X . Assume that A is a non empty open set of X and $|\{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\}| < \aleph_0$. Assume that $H = \{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\}$ is equal to $\{\xi_1, \dots, \xi_n\}$ with $\xi_1 < \xi_2 < \dots < \xi_n$.

If $B = A \cap X_{\alpha_0} \neq \emptyset$, then $A \cap X_{\xi_n} = B$. But A and X_{ξ_n} are open sets in X , so B is a non empty open set in X , contradicting the fact that $\text{int}(X_{\alpha_0}) = \emptyset$. This means that $A \cap X_{\alpha_0}$ must be empty.

Now, let $B = A \cap Y_{\xi_n}$. B is not empty and $A \cap X_{\xi_{n-1}} = B$. Thus, B is a non empty open set in X which does not intersect any member of \mathcal{C}_{ξ_n} . If $\xi_n = \alpha + 1$, \mathcal{C}_{ξ_n} is a maximal cellular collection of open sets contained in $(\bigcup \mathcal{C}_\alpha) \setminus \{a_C^\alpha : C \in \mathcal{C}_\alpha\} = X_\alpha \setminus \{a_C^\alpha : C \in \mathcal{C}_\alpha\}$. Hence, $B \cap \{a_C^\alpha : C \in \mathcal{C}_\alpha\} \neq \emptyset$. Let $a_C^\alpha \in B$. We have that $M = (C \cap B) \setminus \{a_C^\alpha\}$ is an open set contained in $X_\alpha \setminus \{a_C^\alpha : C \in \mathcal{C}_\alpha\}$ and no element in \mathcal{C}_ξ intersects M . By maximality of

\mathcal{C}_ξ , we must have that M is empty; that is, $C \cap B = \{a_C^\gamma\}$, and this is not possible because X does not have isolated points.

Now assume that ξ_n is a limit ordinal. Since B is open and $B \subseteq \bigcap_{\xi < \xi_n} X_\xi$, B must be contained in $\text{int}(\bigcap_{\xi < \xi_n} X_\xi)$. Since $\{a_{\xi_n}\}$ is closed and B does not intersect any element of \mathcal{C}_{ξ_n} which is a maximal cellular family of open sets contained in the set $\text{int}(\bigcap_{\xi < \xi_n} X_\xi) \setminus \{a_{\xi_n}\}$, B must be equal to $\{a_{\xi_n}\}$, which is again a contradiction.

Therefore, $|\{\xi < \alpha_0 : A \cap Y_\xi \neq \emptyset\}|$ must be equal to \aleph_0 . \square

Since the cellularity of a space is a monotone function when it is applied on dense subspaces, and using Theorem 1.3, we conclude:

4.2. Corollary. *[SH] Every T_2 Baire space with $c(X) \leq \aleph_0$ is ω -resolvable.*

Example 4.3 in [25] (see example 2.3 above) gives us a space which is Baire, T_1 with countable cellularity but it is not almost- ω -resolvable. This example is constructed assuming the existence of measurable cardinals. Moreover, there is a model M in which SH holds and there are measurable cardinals. So we cannot get something stronger than our results of this section by assuming only T_1 . Furthermore, we cannot erase the Baire condition in Corollary 4.2 because there is in ZFC a Tychonoff, countable irresolvable space (see Examples 2.5). Finally, in 2.2 we list an example of a space with cellularity $\leq \aleph_1$ which is Baire and is not almost- ω -resolvable. This last example is given by assuming the existence of an ω_1 -complete ideal over ω_1 which has a dense set of cardinality ω_1 . Hence, it is natural to ask:

4.3. Question. *Does MA imply that every crowded T_2 space of cellularity $< \aleph_1$ is almost- ω -resolvable?*

In this question, we cannot change “almost- ω -resolvable” for “resolvable” since there is in ZFC an irresolvable countable space.

5. ALMOST- ω -IRRESOLVABLE SPACES

A space is *almost- ω -irresolvable* if it is not almost- ω -resolvable. In a similar way we define almost irresolvable spaces.

5.1. Proposition. *If X is almost- ω -irresolvable, then there is a non-empty open subset U of X which is hereditarily almost- ω -irresolvable.*

Proof. Let \mathcal{U} be the collection of every subspace Y of X which is almost- ω -resolvable. The set $Z = cl_X(\bigcup \mathcal{U})$ is almost- ω -resolvable and $U = X \setminus Z$ is not empty and satisfies the requirements. \square

5.2. Proposition. *If X is open hereditarily almost- ω -irresolvable, then X is a Baire space.*

Proof. Let $\{U_n : n < \omega\}$ be a sequence of open and dense subsets of X . We can choose this sequence to be \subseteq -decreasing. Denote by F the set $\bigcap_{n < \omega} U_n$. We claim that F is dense in X . In fact, if for a $k < \omega$, $cl_X F \supseteq U_k$, then $cl_X F \supseteq cl_X U_k = X$ and F is dense. Now, assume that for each $n < \omega$, $U_n \setminus cl_X F$ is not empty. In this case, the collection $T = \{i < \omega : (U_i \setminus U_{i+1}) \cap (X \setminus cl_X F) \neq \emptyset\}$ is infinite. For each $i \in T$, we put $T_i = (U_i \setminus U_{i+1}) \cap (X \setminus cl_X F)$. The collection $\{T_i : i < \omega\}$ forms an almost- ω -resolution of $X \setminus cl_X F$. But this is not possible. \square

5.3. Corollary. *If there is an almost resolvable space X which is almost- ω -irresolvable, then there is a resolvable Baire open subspace U of X which is hereditarily almost- ω -irresolvable.*

Proof. Let X be an almost-resolvable almost- ω -irresolvable space. The space X contains a non-empty open subspace U which is hereditarily almost- ω -irresolvable. By Proposition 5.2, U is a Baire space; so, it is resolvable being almost resolvable. \square

5.4. Corollary. *There is an almost resolvable space X which is almost- ω -irresolvable if and only if there is an almost resolvable Baire space which is hereditarily almost- ω -irresolvable.*

As a consequence of the previous result, we have that almost resolvability and almost- ω -resolvability coincide in the class of spaces X in which every open subset is not a Baire space. It was obtained in [2, Corollary 5.21] even more: every space which does not contain a Baire open subspace is almost- ω -resolvable.

5.5. Proposition. *Let X be a T_1 space. Then X is hereditarily resolvable if and only if X is hereditarily ω -resolvable.*

Proof. Let Y be a crowded subspace of X and assume that Y is not ω -resolvable. Then, there is $k \in \omega$ with $k > 1$ such that X is k -resolvable but X is not $(k + 1)$ -resolvable [15]. So there are D_0, \dots, D_{k-1} dense and pairwise disjoint subspaces of Y . But, then, each D_i is crowded and irresolvable, a contradiction. \square

5.6. Proposition. *Let X be with the property that every of its crowded subspaces is Baire. Then X is hereditarily ω -resolvable iff X is hereditarily resolvable iff X is hereditarily almost- ω -resolvable iff X is hereditarily almost resolvable.*

Several results established in [2, Section 5] and [25, Section 4] relate Baire irresolvable spaces with the property of almost- ω -resolvability (see also [1, Section 3]). In the following theorem we obtain the most general possible result in the mood of these propositions.

5.7. Theorem. *For crowded T_1 spaces and for a crowded-hereditarily topological property P , the following assertions are equivalent:*

- (1) every Baire space with P is ω -resolvable,
- (2) every Baire space with P is resolvable,
- (3) every space with P is almost- ω -resolvable,
- (4) every space with P is almost resolvable.

Proof. The implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are evident.

(2) \Rightarrow (3) : Assume that X is not almost- ω -resolvable and satisfies P . The space X contains an open and non-empty subset U which is hereditarily almost- ω -irresolvable. By Proposition 5.5, U is not hereditarily resolvable, so there is a crowded subspace Y which is not resolvable. Observe that Y is hereditarily almost- ω -irresolvable, then Y is an irresolvable Baire space because of Proposition 5.2. Since P is a crowded-hereditarily topological property, Y satisfies P too.

(4) \Rightarrow (2) : Assume that X is a Baire space with P . By hypothesis, X is almost resolvable and every Baire almost resolvable space is resolvable (see [2, Corollary 5.4]).

(3) \Rightarrow (1) : Assume that X is a Baire space with P . By hypothesis, every crowded subspace Y of X has P and so it is almost- ω -resolvable; hence X is ω -resolvable because of Theorem 1.3. \square

Taking P equal to “ X is a crowded topological space”, we have:

5.8. Corollary. *For crowded T_1 spaces, the following assertions are equivalent:*

- (1) every Baire space is ω -resolvable,
- (2) every Baire space is resolvable,
- (3) every space is almost- ω -resolvable,
- (4) every space is almost resolvable.

A space is *locally homogeneous* if each of its points has a homogeneous neighborhood. For a cardinal number $\kappa \geq 1$, we will say that X is *exactly κ -resolvable*, in symbols $E_\kappa R$, if X is κ -resolvable but is not κ^+ -resolvable. The space X is said to be $OE_\kappa R$ if every nonempty open set in X is $E_\kappa R$. The concept and examples of $E_n R$ spaces for $n \in \omega$ have existed in the literature for some time (see, for example, [10] and [8]). It is clear that the $OE_\kappa R$ spaces are $E_\kappa R$. The above definitions can be viewed as natural generalizations of the concepts of irresolvable and open-hereditarily irresolvable spaces since $E_1 R$ and irresolvability are the same concept and $OE_1 R$ and open-hereditarily irresolvability coincide.

It was proved in [1, Theorem 3.13] that every locally homogeneous irresolvable space such that its cardinality is not a measurable cardinal is of the first category. Also, Li Feng and O. Masaveu [13] proved that every crowded topological space X can be written as

$$X = \Omega \cup cl_X \left(\bigcup_{n=1}^{\infty} O_n \right),$$

where

- (1) for each n , O_n is an open, possibly empty, subset of X ;
- (2) for each n , if $O_n \neq \emptyset$, then it is $OE_n R$;
- (3) for $n \neq m$, $O_n \cap O_m = \emptyset$; and
- (4) Ω is an open, possibly empty, ω -resolvable subset of X .

Thus we obtain the following:

5.9. Proposition. *Every locally homogeneous Baire space of cardinality strictly less than the first measurable cardinal is resolvable.*

Proof. Let X be a locally homogeneous Baire space. Write X as Feng and Masaveu say: $X = \Omega \cup cl_X \left(\bigcup_{n=1}^{\infty} O_n \right)$. Assume that O_1 is not empty and take $x \in O_1$. There is a homogeneous neighborhood W of x . (Observe that W has to be contained in $X \setminus cl_X \left(\Omega \cup \bigcup_{n>1} O_n \right) \subseteq int_X cl_X O_1$). On the other hand, O_1 is open hereditarily irresolvable, so $int_X W \cap O_1$ is irresolvable. Since $int_X W \cap O_1$ is a non-empty open subset of W , W is irresolvable. By Theorem 3.13 in [1], W is of first category. In particular the open and nonempty subset $O_1 \cap int_X W$ of X is of first category in itself, but this is not possible because X is a Baire space. Hence, $O_1 = \emptyset$ and X is resolvable. \square

5.10. Questions. (1) *Is every pseudocompact (resp., Čech-complete) Tychonoff space almost- ω -resolvable in ZFC?*

(2) *Is every Baire locally homogeneous (resp., homogeneous, topological group) ω -resolvable?*

(3) *For each $n > 1$, is there a Baire $OE_n R$ space?*

6. THE INFINITE π -NETWEIGHT AND $Seq(u_t)$ SPACES

We define the *infinite π -networkweight* of a crowded space X , $\pi nw^*(X)$, as the minimum infinite cardinal of a π -network with infinite elements. And $\pi nw(X)$ is the minimum infinite

cardinal of a π -network in X . It is easy to prove that $\pi nw(X) = d(X)$ for every topological space X . Moreover, for a crowded space X , we have $d(X) \leq \pi nw^*(X) \leq \min\{d(X) \cdot R(X), \pi w(X)\}$, where $R(X)$ was defined before Corollary 3.12. Besides, for every metrizable space X we have $d(X) = w(X)$. So, for a crowded metrizable space X , the equality $\pi nw^*(X) = \pi nw(X)$ always holds. We have the same phenomenon for spaces of the form $C_p(X)$, the space of real continuous function defined on X with the pointwise convergent topology (here, X is not necessarily crowded). Indeed, for $f \in C_p(X)$, the sequence $(f_n)_{n < \omega}$ where $f_n = f + 1/n$, converges to f . So, if D is a dense subset of $C_p(X)$ with cardinality equal to $d(C_p(X))$, the collection $\{\{f\} \cup \{f_i : i \geq n\} : f \in D, n < \omega\}$ is a π -network of cardinality $d(C_p(X))$ constituted by infinite elements. So, $\pi nw^*(C_p(X)) = \pi nw(C_p(X))$. In particular, for every cardinal number κ , $\pi nw^*(\mathbb{R}^\kappa) = d(\mathbb{R}^\kappa)$. The same can be say for spaces of the form $C_p(X, 2)$ where X is an infinite zero-dimensional T_2 space. In fact, we can take an infinite discrete subspace $Y = \{x_n : n < \omega\}$ of X , and clopen subsets $\{V_n : n < \omega\}$ such that, for each $n < \omega$, $Y \cap V_n = \{x_n\}$. The characteristic functions χ_{V_n} constitute a sequence which converge to the constant function 0. So, in this case too, $\pi nw^*(C_p(X, 2)) = d(C_p(X, 2))$.

We have already mentioned that in [1] was constructed in *ZFC* a dense countable Y of 2^ω which is irresolvable. This space has $\pi nw(Y) = \aleph_0$, but every of its countable π -networks has to have finite elements, because otherwise Y would be maximally resolvable (see Theorem 2.8.1). The *Seq*(u_t) spaces considered below are also examples of spaces of this kind.

We recall that for a $p \in \omega^*$, $\chi(p) = \min\{|b| : b \text{ is a base for } p\}$. Of cours we can also define: $\pi\chi(p) = \min\{|b| : b \text{ is a } \pi\text{-base for } p\}$ where a family of infinite sets \mathcal{G} in ω is a π -base for p if every member of p contain an element of \mathcal{G} . It is not difficult to prove that for every $p \in \omega^*$, $\pi\chi(p) \leq \chi(p)$ and $\pi\chi(p) > \aleph_0$. In fact, assume that N_0, \dots, N_k, \dots are infinite subsets of ω . By recurition, we can construct two sequences $A = \{a_0, \dots, a_n, \dots\}$ and $B = \{b_0, \dots, b_n, \dots\}$ such that the elements in $A \cup B$ are pairwise diferent, and for each $n < \omega$, $a_n, b_n \in N_n$. If $A \in p$ then A is an element of p which does not contain any N_k . If $A \notin p$, then $\omega \setminus A$ belongs to p and does not contain any N_k .

By *Seq* we mean the set of all finite sequences of natural numbers. More precisely, for each natural number $n \in \omega$, let ${}^n\omega = \{t : t \text{ is a function and } t : n \rightarrow \omega\}$. Then $Seq = \bigcup_{n \in \omega} {}^n\omega$. If $t \in Seq$, with domain $k = \{0, 1, \dots, (k-1)\}$, and $n \in \omega$, let $t \frown n$ denote the function $t \cup \{(k, n)\}$. For every $t \in Seq$ let u_t be a non-principal ultrafilter on ω . By $Seq(\{u_t : t \in Seq\})$ we denote the space with underlying set *Seq* and topology defined by declaring a set $U \subseteq Seq$ to be open if and only if

$$(\forall t \in U)\{n \in \omega : t \frown n \in U\} \in u_t.$$

For short, we write $Seq(u_t)$ instead of $Seq(\{u_t : t \in Seq\})$. We also consider the case where there is a single non-principal ultrafilter p in ω such that $u_t = p$ for all $t \in Seq$, and in this case we write $Seq(p)$ instead of $Seq(u_t)$.

We use the following notation of W. Lindgren and A. Szymanski [19]; put $L_n = \{s \in Seq : \text{dom}(s) = n\}$, and for any $s \in Seq$ the *cone over s* is defined by $C(s) = \{t \in Seq : s \subseteq t\}$. In particular, $L_0 = \{\emptyset\}$. We add some other notations: For each $s \in L_n$, $T(s) = \{t \in L_{n+1} : s \subseteq t\}$. Observe that for every $s \in Seq$, $C(s)$ is a clopen subset of $Seq(u_t)$.

It is well-known that for any choice of $\{u_t : t \in Seq\} \subseteq \omega^*$, the space $Seq(u_t)$ is a zero-dimensional, extremally disconnected, Hausdorff space with no isolated points. By the way, $Seq(p)$ is homogeneous and if p is Ramsey, there is a binary group operation $+$ such that $(Seq(p), +)$ is a topological group (see [26]).

6.1. Proposition. *Every $Seq(u_t)$ space is ω -resolvable.*

Proof. In fact, let $\{E_n : n < \omega\}$ be a partition of ω where each E_n is infinite. Set $D_n = \bigcup_{i \in E_n} L_i$. Each D_n is dense in $Seq(u_t)$ and $D_n \cap D_m = \emptyset$ if $n \neq m$. \square

6.2. Proposition. *Let $\{u_t : t \in Seq\} \subseteq \omega^*$. Then, the infinite π -netweight of $Seq(u_t)$ is not countable.*

Proof. For each $n < \omega$, each $s \in L_n$, and each sequence S of subcollections of the form

$$\{B(s)\}, \{B(s, i_{n+1}) : i_{n+1} \in B(s)\}, \{B(s, i_{n+1}, i_{n+2}) : i_{n+1} \in B(s), i_{n+2} \in B(s, i_{n+1})\}, \dots,$$

$$\{B(s, i_{n+1}, \dots, i_{n+k+1}) : i_{n+1} \in B(s), i_{n+1} \in B(s, i_{n+1}), \dots, i_{n+k+1} \in B(s, i_{n+1}, \dots, i_{n+k})\}, \dots$$

where $B(s) \in u_s$ and, if $i_{n+1} \in B(s), i_{n+2} \in B(s, i_{n+1}), \dots, i_{n+k} \in B(s, i_{n+1}, \dots, i_{n+k-1}), B(s, i_{n+1}, \dots, i_{n+k}) \in u_t$ with $t = s \widehat{\cap} i_{n+1} \dots \widehat{\cap} i_{n+k}$, we define a set $V(s, S)$ as follows:

$$V(s, S) = \{s\} \cup \{t \in Seq(p) : m \in \omega, t \in L_{n+m+1}, s \subseteq t, t(n+1) \in B(s),$$

$$t(n+2) \in B(s, t(n+1)), \dots, t(n+m+1) \in B(s, t(n+1), t(n+2), \dots, t(n+m))\}.$$

We call this set $V(s, S)$ *cascade of $Seq(p)$ defined by (s, S)* . Moreover, we will called each sequence S , described as above, *fan on $(s, (u_t))$* .

Of cours the collection of cascades form a base of clopen sets for $Seq(u_t)$.

Claim 1. If $\mathcal{N} = \{N_0, \dots, N_k, \dots\}$ is a countable set of infinite subsets of $Seq(u_t)$, then \mathcal{N} is not a π -network of $Seq(u_t)$.

We are going to prove Claim 1 in several lemmas.

Claim 1.1. If \mathcal{M} is a finite collection of subsets of Seq , then there is a non empty open set A of $Seq(u_t)$ such that $M \setminus A \neq \emptyset$ for all $M \in \mathcal{M}$.

Proof. Take s_0, \dots, s_n elements in Seq such that each M in \mathcal{M} contains one of this points. There is $k < \omega$ such that $s_i \in L_m$ implies $m < k$ for all $i \in \{0, \dots, n\}$. Take $s \in L_k$. The cone $C(s)$ is open and contains no element in \mathcal{M} .

Claim 1.2. Assume that $F \subseteq Seq(u_t)$ is such that $|F \cap T(s)| \leq 1$ for every $s \in Seq$. Then, F is a proper closed subset of $Seq(u_t)$.

Proof. Let P be the set $\{s < Seq : F \cap T(s) \neq \emptyset\}$. Let z_s be the only point belonging to $F \cap T(s)$ for each $s \in P$. Let $x \in Seq(u_t) \setminus F$. Assume that $x = (n_0, \dots, n_k)$ (the argument is similar if $x = \emptyset$). Let

$$S = \{\{B(x)\}, \{B(x, i_0) : i_0 \in B(x)\}, \{B(x, i_0, i_1) : i_0 \in B(x), i_1 \in B(x, i_0)\}, \dots,$$

$$\{B(x, i_0, \dots, i_{k+1}) : i_0 \in B(x), i_1 \in B(x, i_1), \dots, i_{k+1} \in B(x, i_0, \dots, i_k)\}, \dots\}$$

be a fan on $(x, (u_t))$. We claim that the set $V(x, S) \setminus F$ is an open set. Indeed, if $y \in V(x, S) \setminus F$, y is of the form $(n_0, \dots, n_k, i_0, \dots, i_{m+1})$ where $m < \omega$, $i_0 \in B(x), i_1 \in B(x, i_0), \dots, i_{m+1} \in B(x, i_0, i_1, \dots, i_m)$.

The set $\{l < \omega : (n_0, \dots, n_k, i_0, \dots, i_{m+1}, l) \in V(x, S) \setminus F\}$ is equal to

$$B(x, i_0, i_1, \dots, i_{m+1}) \setminus F.$$

Moreover, the set $B(x, i_0, i_1, \dots, i_{m+1}) \cap F = G$ is either empty if $F \cap T(x, i_0, i_1, \dots, i_{m+1}) = \emptyset$, or $G = \{z_{(x, i_0, i_1, \dots, i_{m+1})}\}$ if $F \cap T(x, i_0, i_1, \dots, i_{m+1}) \neq \emptyset$. Of course, in both cases, $B(x, i_0, i_1, \dots, i_{m+1}) \setminus F$ belongs to u_t where $t = x \hat{\ } i_0 \hat{\ } \dots \hat{\ } i_{m+1}$. This means that $V(x, s) \setminus F$ is open.

Claim 1.3. Let $\mathcal{M} = \{N \in \mathcal{N} : \forall s \in \text{Seq}(|N \cap T(s)| < \aleph_0)\}$. Then, there is a non empty open set A of $\text{Seq}(u_t)$ such that $N \setminus A \neq \emptyset$ for all $N \in \mathcal{M}$.

Proof. First, we define in Seq a well order \sqsubseteq as follows: \emptyset is the \sqsubseteq -first element, and for two elements s and t different to \emptyset , we define $s \sqsubseteq t$ if either $s \in L_{n+1}$, $t \in L_{m+1}$ and $n < m$, or $n = m$ and $s(n) < t(n)$.

Because of Claim 1.1, we can assume that \mathcal{M} is infinite. We faithfully enumerate \mathcal{M} as $\{M_0, M_1, \dots, M_k, \dots\}$. Consider the set $J = \{s \in \text{Seq} : \exists M \in \mathcal{M} \text{ such that } T(s) \cap M \neq \emptyset\}$. Because of the definition of \mathcal{M} , we must have $|J| = \aleph_0$. Hence, we can enumerate J as $\{s_m : m < \omega\}$ in such a way that $s_0 \sqsubseteq s_1 \sqsubseteq \dots \sqsubseteq s_n \sqsubseteq s_{n+1} \sqsubseteq \dots$.

Let k_0 be the first natural number m such that $M_m \cap T(s_0) \neq \emptyset$. We take $z_0 \in M_{k_0} \cap T(s_0)$. Assume that we have already defined two finite sequences k_0, \dots, k_l and z_0, \dots, z_l such that

- (1) for each $i \in \{0, \dots, l-1\}$, k_{i+1} is the first natural number $m \in \omega \setminus \{k_0, \dots, k_i\}$ such that $M_m \cap T(s_{i+1}) \neq \emptyset$, and
- (2) $z_{i+1} \in M_{k_{i+1}} \cap T(s_{i+1})$ for each $i \in \{0, \dots, l-1\}$.

We define now k_{l+1} as the first natural number $m \in \omega \setminus \{k_0, \dots, k_l\}$ such that $M_m \cap T(s_{l+1}) \neq \emptyset$. Take $z_{l+1} \in M_{k_{l+1}} \cap T(s_{l+1})$.

Observe that $\{k_i : i < \omega\} = \omega$. Indeed, assume that $\{0, \dots, m\} \subseteq \{k_i : i < \omega\}$ and $\{k_{i_0}, \dots, k_{i_m}\} = \{0, \dots, m\}$. Let j be a natural number greater than k_{i_l} for all $l \in \{0, \dots, m\}$ and such that $M_{m+1} \cap T(s_j) \neq \emptyset$. Then we must have $m+1 \in \{k_0, \dots, k_j\}$.

We put $F = \{z_i : i < \omega\}$. The set F satisfies the conditions required in Claim 1.2; so, F is a proper closed subset of $\text{Seq}(u_t)$. Therefore, $A = \text{Seq}(u_t) \setminus F$ is a non-empty open set which does not contain any of the sets $M \in \mathcal{M}$.

Claim 1.4. Let $\mathcal{O} = \mathcal{N} \setminus \mathcal{M} = \{N \in \mathcal{N} : \exists s \in \text{Seq}(|N \cap T(s)| \geq \aleph_0)\}$. Then, there is an open set B of $\text{Seq}(u_t)$ such that $N \setminus B \neq \emptyset$ for all $N \in \mathcal{O}$.

Proof. Let $T = \{n < \omega : N_n \in \mathcal{O}\}$. The open set B will be an open cascade $V(s, S)$ defined by (s, S) where $s = \emptyset$ and the fan

$$\begin{aligned} S = \{ & \{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), \\ & i_2 \in B(s, i_1)\}, \dots, \{B(s, i_1, \dots, i_{k+1}) : i_1 \in B(s), i_1 \in \\ & B(s, i_1), \dots, i_{k+1} \in B(s, i_1, \dots, i_k)\}, \dots \} \end{aligned}$$

will be constructed by recursion.

Assume that we have already selected

$$\begin{aligned} \{ & \{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), \\ & i_2 \in B(s, i_1)\}, \dots, \{B(s, i_1, \dots, i_k) : i_1 \in B(s), i_2 \in \\ & B(s, i_1), \dots, i_k \in B(s, i_1, \dots, i_{k-1})\} \}. \end{aligned}$$

For each sequence $i_1 \in B(s), i_2 \in B(s, i_1), \dots, i_{k+1} \in B(s, i_1, i_2, \dots, i_k)$, consider the ultrafilter u_t where $t = s \hat{\ } i_1 \hat{\ } \dots \hat{\ } i_k$, and consider the set $P(s, i_1, \dots, i_{k+1}) = \{n \in T :$

$|N_n \cap T(s, i_1, \dots, i_k)| \geq \aleph_0\}$. If $P(s, i_1, \dots, i_{k+1})$ is empty, we choose $B(s, i_1, \dots, i_{k+1})$ to be an arbitrary element of u_t . If $P(s, i_1, \dots, i_{k+1})$ is not empty, there is $B(s, i_1, \dots, i_{k+1}) \in u_t$ such that $N_n \setminus B(s, i_1, \dots, i_{k+1}) \neq \emptyset$ for every $n \in P(s, i_1, \dots, i_{k+1})$ because $\pi\chi(u_t) > \aleph_0$.

We have already finished the description of the recursive process that define the fan S . The set $B = V(s, S)$ is the required open set.

We finished the proof of Claim 1 by saying that the open set $A \cap B$, where A was defined in the proof of Claim 1.3 and B in that of Claim 1.4, is not empty and does not contain any of the elements in \mathcal{N} . \square

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