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Generalized independence

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ABSTRACT

We explore different generalizations of the classical concept of independent families on ω following the study initiated by Kunen, Fischer, Eskew and Montoya. We show that under $(\mathrm{D}\ell)^*_\kappa$ we can get strongly κ-independent families of size 2^κ and present an equivalence of GCH in terms of strongly independent families. We merge the two natural ways of generalizing independent families through a filter or an ideal and we focus on the $\mathcal{C}$-independent families, where $\mathcal{C}$ is the club filter. Also we show a relationship between the existence of $\mathcal{J}$-independent families and the saturation of the ideal $\mathcal{J}$.

0. Introduction

Independent families are objects with strong combinatorial properties. Since their appearance in [2] and [6], these families have been related to many other objects, such as almost disjoint families, ultrafilters and ideals. See for example [4].

Independent families are naturally defined over the set of non-negative integers ω; however, it is not clear what their natural generalization to larger cardinals should be. An *independent family* on ω is a family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that if $S, T \subseteq \mathcal{I}$ are finite and disjoint subfamilies then $\bigcap S \setminus \bigcup T$ is infinite (we call this set a finite Boolean combination from $\mathcal{I}$). In other words, on ω, a family is independent if all its finite Boolean combinations are infinite. When we move to the case of an arbitrary cardinal κ the notion

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of independence could be generalized in at least two different ways: the first would be by allowing larger Boolean combinations, that is, not only finite Boolean combinations but also the ones of length less than or equal to λ for some given λ and the second way would be to ask that finite Boolean combinations not only have infinite cardinality (or cardinality κ) but that they fulfill some notion of *largeness*.

The first of these generalizations that we are aware of was studied by Kenneth Kunen [11]. Among other things, he proved the existence of maximal σ-independent families of subsets of some cardinal $\chi > \omega_1$ and he proved that the existence of such families is equiconsistent with the existence of a measurable cardinal. A more recent study of this kind of generalizations was started by Vera Fischer and Diana Montoya in [3] and then continued by Monroe Eskew and Vera Fischer in [1]. In this last referred work, the authors study higher analogous of the classical notion of maximal independent family on ω. As in some other works, they point out that the Axiom of Choice does imply maximal independent families exists as long as the Boolean combinations under consideration are finite; however, the panorama is totally different if we consider longer Boolean combinations. For instance, they show, in addition to other very remarkable results, that if $\mathbb{P}$ is a nontrivial forcing either of size less than κ or satisfying the ν-cc for some $\nu < \kappa$, then $\mathbb{P}$ forces that there are no maximal strongly κ-independent families. On the other hand, if κ is a supercompact cardinal, then there is a forcing extension in which for all κ-directed-closed posets $\mathbb{P}$ that force $2^\kappa < \kappa^{+\omega}$, $\mathbb{P}$ forces that there are maximal strongly κ-independent families.

In the first section we define strongly κ-independent families, we justify the reason for considering Boolean combinations of length less than κ and we give a characterization of the Continuum Hypothesis in terms of the existence of one of these families for $\kappa = \omega_1$, even more, we show that $2^\kappa = \kappa^+$ is equivalent to the existence of a certain strongly κ^+-independent family (see Theorem 1.8).

Perhaps the most important result of section one is the fact that the existence of a $(\mathrm{D}\ell)^*_\kappa$-sequence implies the existence of a strongly κ-independent family of maximum size. With the help of Shelah's principle $(\mathrm{D}\ell)^*_\kappa$ we can offer a wide variety of cardinals for which the existence of a strongly κ-independent family was unknown.

In this section we also show a relationship between the existence of some of these families and the existence of a strongly inaccessible cardinal.

In the second section, we study a second generalization of independent families, what we have called $\mathcal{F}$-independent or $\mathcal{J}$-independent families, depending on whether $\mathcal{F}$ is a filter or $\mathcal{J}$ is an ideal on a given cardinal κ. We say that a family is $\mathcal{F}$-independent (or $\mathcal{J}$-independent) if every finite Boolean combination is in $\mathcal{F}^+$ (or in $\mathcal{J}^+$ respectively). For a filter $\mathcal{F}$ some conditions on it are shown so that there are $\mathcal{F}$-independent families; in this same direction we show that strongly $\mathcal{F}$-independent families can also exist, i.e., a generalization in two senses of classical independent families. Later we will focus on the *club* filter, closed and unbounded sets, and show some similarities between this new notion of independence and the classical one. Finally, for an ideal $\mathcal{J} \subseteq \mathcal{P}(\kappa)$, we show that exists a relationship between the existence (or non-existence) of $\mathcal{J}$-independent families and the saturation of $\mathcal{J}$, therefore with some properties of the cardinal κ.

Throughout the article we talk about families with a certain property of independence and we also talk about ideals, we will use the font $\mathcal{I}$ and $\mathcal{J}$ to denote independent families (or so) and the font $\mathfrak{I}$ or $\mathfrak{J}$ to denote ideals.

1. Strongly independent families

For a cardinal κ and $A \subseteq \kappa$, we will use the usual notation A^0 to denote A and A^1 to denote $\kappa \setminus A$. If X and Y are sets and s is a function, we will use the notation $s; X \to Y$ to express that s is a partial function[2] from X to Y, i.e., $dom(s) \subseteq X$ and s takes its values on Y. For a given set $\mathcal{I}$ and a cardinal λ, we will

[2] Note the semicolon instead of the colon.

denote the collection of partial functions from $\mathcal{I}$ to $2 = \{0, 1\}$ of size less that λ, $\{s; \mathcal{I} \to 2 : |s| < \lambda\}$, by $\mathrm{FF}_{<\lambda}(\mathcal{I})$.

The rest of the terminology is canonical and it is the one followed by current literature in set theory.

Definition 1.1. If $\mathcal{I}$ is a family of subsets of a cardinal κ and $h; \mathcal{I} \to 2$, then $\mathcal{I}^h = \bigcap_{I \in dom(h)} I^{h(I)}$ is *the Boolean combination of* $\mathcal{I}$ *determined by* h. If h is finite then we say that $\mathcal{I}^h$ is a finite Boolean combination. If h has cardinality λ we say that $\mathcal{I}^h$ is a *Boolean combination of length* λ.

Given a cardinal μ, the set whose elements are all Boolean combinations from $\mathcal{I}$ of length less than μ is the μ-*envelope* of $\mathcal{I}$ and we denote it by $\mathrm{ENV}_{<\mu}(\mathcal{I})$.

Definition 1.2. Let μ and κ be infinite cardinals with $\mu \leq \kappa$ and let $\mathcal{I} \subseteq \mathcal{P}(\kappa)$. Then $\mathcal{I}$ is (μ, κ)-*independent* if every Boolean combination of length less than μ of elements of $\mathcal{I}$ has size κ, i.e., $\mathrm{ENV}_{<\mu}(\mathcal{I}) \subseteq [\kappa]^\kappa$.

As mentioned in the introduction, Kunen was the first to study systematically (μ, κ)-independent families. He obtained some very deep results, one of which is the following:

Theorem 1.3. *[11] The existence of maximal* $(\omega_1, 2^{\omega_1})$*-independent family is equiconsistent with existence of a measurable cardinal.*[3]

Moreover, Kunen proved that if λ is strongly compact, then there is a forcing extension such that there is a maximal (ω_1, χ)-independent family for all $\chi \geq \lambda$ such that $\mathrm{cof}(\chi) \geq \lambda$. Although this result is highly remarkable, note that the existence of a maximal $(\omega_1, 2^{\omega_1})$-independent family is not directly related, at least not in an obvious manner, to the existence of a (ω_1, ω_1)-independent family. In general, the existence of a maximal (μ, κ)-independent family with $\mu < \kappa$ does not seem obviously related to the existence of (μ, μ)-independent families or (κ, κ)-independent families. Furthermore, while it is true that (μ, μ)-independent families are implicitly mentioned in Kunen's paper, in general, most of his constructions of (μ, κ)-independent families are such that $\mu < \kappa$; consequently, some portion of his work does not allude to (μ, μ)-independent families, which are the families that will interest us the most in this article.

In order to shorten the notation, we will employ the one used in [3], where κ-*independent* abbreviates (ω, κ)-independent and *strongly* κ-*independent* abbreviates (κ, κ)-independent. Note that with the convention we have just adopted, what we have called an independent family in the introduction turns out to be a ω-independent family.

Normally, after definitions, examples come; instead we now present a typical example of the classical case of an ω-independent family. Latter we shall use it to give examples of the generalizations introduced in Definition 1.2.

Example 1.4. Let p_n be the n-th prime number and $C_n = \{mp_n : m \in \omega\}$. The family $\mathcal{I} = \{C_n : n \in \omega\} \subseteq [\omega]^\omega$ is ω-independent.

The family in the previous example is a ω-independent family such that $\mathcal{I}^h = \emptyset$ for any infinite Boolean combination $h; \omega \to 2$ such that $h^{-1}[\{0\}]$ is infinite. Nevertheless, this does not mean that this family is not strongly ω-independent, since in the case of $\kappa = \omega$, κ-independence and strongly κ-independence agree (it is also the unique cardinal κ where they do). It is easy to observe that for any infinite ω-independent family $\mathcal{I}$ there exists $h; \mathcal{I} \to 2$ infinite such that $\mathcal{I}^h = \emptyset$. In general, we restrict ourselves to Boolean combinations of length less than κ because if $\mathcal{I}$ is a κ-independent family of cardinality at least κ, there is $h; \mathcal{I} \to 2$, with $|h| = \kappa$, such that $\mathcal{I}^h = \emptyset$.

[3] This is Theorem 2 in [11], although a different notation is used there.

The following question naturally arises: For which cardinals κ does there exist (or may exist) a strongly κ-independent family, and for which ones do there exist *large* strongly κ-independent families, that is, of cardinality 2^κ? Fischer and Montoya in [3] provided a partial answer to this question, which has inspired us to use a guessing principle to construct strongly κ-independent families.

Definition 1.5. [14] Let κ be a cardinal. We say that a sequence $\langle S_\alpha : \alpha \in \kappa \rangle$ is a $(\mathrm{D}\ell)^*_\kappa$*-sequence* if:

(1) For every $\alpha \in \kappa$, we have that $S_\alpha \subseteq \mathcal{P}(\alpha)$ and $|S_\alpha| < \kappa$.
(2) For every $X \subseteq \kappa$, the set $\{\alpha \in \kappa : X \cap \alpha \in S_\alpha\}$ is club in κ.

The existence of a $(\mathrm{D}\ell)^*_\kappa$-sequence will be denoted simply as $(\mathrm{D}\ell)^*_\kappa$.

Since the principle $(\mathrm{D}\ell)^*_\kappa$ is not very well known, we decide to include the proof of the next proposition, even though it is essentially a modification of the classical result that $\diamondsuit$ implies CH.

Proposition 1.6. *Let κ and λ be cardinals such that $\lambda < \kappa$ and κ is regular. Then $(\mathrm{D}\ell)^*_\kappa$ implies $2^\lambda \leq \kappa$.*

Proof. Note that if $X \subseteq \lambda$ and $\alpha > \lambda$, then $X \cap \alpha = X$, thus, as $\{\alpha \in \kappa : X \cap \alpha \in S_\alpha\}$ is club, $C_X = \{\alpha \in \kappa : X \in S_\alpha\}$ is too, in particular $C_X \neq \emptyset$. Now note that:

$$|\bigcup_{\lambda<\alpha<\kappa} S_\alpha| = \sum_{\lambda<\alpha<\kappa} |S_\alpha| = \kappa,$$

but by the previous, every $X \subseteq \lambda$ satisfies that $X \in \bigcup_{\lambda<\alpha<\kappa} S_\alpha$, hence $\mathcal{P}(\lambda) \subseteq \bigcup_{\lambda<\alpha<\kappa} S_\alpha$ and consequently $2^\lambda \leq \kappa$. $\square$

With the help of the principle $(\mathrm{D}\ell)^*_\kappa$ we show the possibility of having κ-strongly independent families. As we remark in the introduction, this may provide a wider variety of cardinals where κ-strongly independent families exist.

Proposition 1.7. *Let κ be an uncountable regular cardinal. Then $(\mathrm{D}\ell)^*_\kappa$ implies the existence of a strongly κ-independent family of cardinality 2^κ.*

Proof. Let $\langle S_\alpha : \alpha \in \kappa \rangle$ be a $(\mathrm{D}\ell)^*_\kappa$ sequence and let C be defined as follows:

$$C = \{\langle \gamma, A \rangle : \gamma \in \kappa \wedge A \subseteq S_\gamma\}.$$

Since $|S_\alpha| < \kappa$ for every $\alpha \in \kappa$, by Proposition 1.6,

$$|C| = \sum_{\alpha\in\kappa} 2^{|S_\alpha|} \leq \sum_{\alpha\in\kappa} \kappa = \kappa$$

and it is also clear that $\kappa \leq |C|$, we conclude that $|C| = \kappa$. Thus constructing a strongly κ-independent family can be done with subsets of C (with Boolean combinations computed in C).

For every $X \subseteq \kappa$ let Y_X be defined as follows:

$$Y_X = \{(\gamma, A) \in C : X \cap \gamma \in A\}.$$

Aiming to prove that $\mathcal{I} = \{Y_X : X \subseteq \kappa\}$ is strongly κ-independent, set $\{X_i : i \in I_0\}, \{Z_j : j \in I_1\} \subseteq \mathcal{P}(\kappa)$ two disjoint collections, with $|I_0|, |I_1| < \kappa$.

For every pair $i, i' \in I_0$ with $i \neq i'$ let $\gamma_{i,i'} \in \kappa$ be such that

$$X_i \cap \gamma_{i,i'} \neq X_{i'} \cap \gamma_{i,i'}.$$

Observe that if $\gamma \geq \gamma_{i,i'}$ then $X_i \cap \gamma \neq X_{i'} \cap \gamma$; analogously for $j, j' \in I_1$, with $j \neq j'$ let $\alpha_{j,j'}$ be such that

$$Z_j \cap \alpha_{j,j'} \neq Z_{j'} \cap \alpha_{j,j'}.$$

Finally if $i \in I_0$ and $j \in I_1$, let $\beta_{i,j} \in \kappa$ be such that

$$X_i \cap \beta_{i,j} \neq Z_j \cap \beta_{i,j}.$$

If we define $B \subseteq \kappa$ as:

$$B = \{\gamma_{i,i'} : i, i' \in I_0 \wedge i \neq i'\} \cup \{\gamma_{j,j'} : j, j' \in I_1 \wedge j \neq j'\} \cup \{\gamma_{i,j} : i \in I_0 \wedge j \in I_1\},$$

it is clear that $|B| < \kappa$ and, as κ is regular, there exists $\gamma_0 \in \kappa$ such that B is bounded by γ_0. Now, if $\gamma \in \kappa$ is larger that γ_0, then this one satisfies the following:

(1) $X_i \cap \gamma \neq X_{i'} \cap \gamma$ if $i, i' \in I_0$ with $i \neq i'$.
(2) $Z_j \cap \gamma \neq Z_{j'} \cap \gamma$ if $j, j' \in I_1$ with $j \neq j'$.
(3) $X_i \cap \gamma \neq Z_j \cap \gamma$ if $i \in I_0$ with $j \in I_1$.

For every $i \in I_0$, consider $D_i = \{\gamma \in \kappa : X_i \cap \gamma \in S_\gamma\}$, which is a club, now put $D = \bigcap_{i \in I_0} D_i$ and let $\gamma \in D$ such that $\gamma > \gamma_0$.

Let $A_\gamma \subseteq S_\gamma$ be defined as:

$$A_\gamma = \{X_i \cap \gamma : i \in I_0\}.$$

So we have that $(\gamma, A_\gamma) \in Y_{X_i}$ for every $i \in I_0$ and $(\gamma, A_\gamma) \notin Y_{Z_j}$ for every $j \in I_1$. This proves that:

$$(\gamma, A_\gamma) \in \bigcap_{i \in I_0} Y_{X_i} \setminus \bigcup_{j \in I_1} Y_{Z_j}$$

and as this happens for every $\gamma \in D$ such that $\gamma > \gamma_0$, then:

$$\Big|\bigcap_{i \in I_0} Y_{X_i} \setminus \bigcup_{j \in I_1} Y_{Z_j}\Big| = \kappa,$$

which finishes the proof. □

If κ is strongly inaccessible then $\langle \mathcal{P}(\alpha) : \alpha \in \kappa \rangle$ turns out to be a $(\mathrm{D}\ell)^*_\kappa$-sequence, hence the previous theorem in particular implies that for every strongly inaccessible cardinal κ there is a large strongly κ-independent family, which is a result obtained by Fischer and Montoya in [3] and which proof is in turn inspired by Hausdorff's original proof that there are ω-independent families of size $\mathfrak{c}$ [6].

In a former version of this paper, in order to obtain previous proposition, we used the well known $\diamondsuit^*(\kappa)$. By a famous theorem of Jensen [8], under $\mathbf{V} = \mathbf{L}$, the principle $\diamondsuit^*(\kappa)$ holds on every successor cardinal. We thank Assaf Rinot for pointing us Shelah's paper [14], so we realized we actually offer a slightly wider spectrum of cardinals for which there are consistently (large) strongly independent families on them without rallying on a very strong hypothesis as $\mathbf{V} = \mathbf{L}$. For instance, in [13], H. Mildenberger and S. Shelah, in their

Fact 2.9, proved that $\diamondsuit^*(\kappa)$ is equivalent to $(\mathrm{D}\ell)^*_\kappa$ for successor cardinals. However, R. Jensen and K. Kunen show in [9] that for limit cardinals, they are not equivalent. This is because if κ is ineffable (in particular, if κ is measurable), then $\diamondsuit^*(\kappa)$ fails, while $(\mathrm{D}\ell)^*_\kappa$ holds due to κ being strongly inaccessible.

On the other hand, the existence of strongly κ-independent families, where κ is a successor cardinal, is also closely related to the Generalized Continuum Hypothesis.

Theorem 1.8. *Let κ be an infinite cardinal. The following two conditions are equivalent.*

(1) There is a strongly κ^+-independent family of cardinality κ.
(2) The equality $2^\kappa = \kappa^+$ is true.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{I} = \{X_\alpha : \alpha \in \kappa\}$ a strongly κ^+-independent family. For all $h \in 2^\kappa$ we have that $\mathcal{I}^h$ has cardinality κ^+ and it is clear that if $h, g \in 2^\kappa$ are different then $\mathcal{I}^h$ and $\mathcal{I}^g$ are disjoint. For every $h \in 2^\kappa$, let $x_h \in \mathcal{I}^h$; then the set $\{x_h : h \in 2^\kappa\}$ is a subset of κ^+ and has cardinality 2^κ, so $2^\kappa \leq \kappa^+$ and therefore $2^\kappa = \kappa^+$.

(2) $\Rightarrow$ (1). Let $f : \kappa^+ \to 2^\kappa \times \kappa^+$ be a bijection (considering 2^κ as the set of all functions from κ to 2). For every $h \in 2^\kappa$, let $X_h = f^{-1}(\{h\} \times \kappa^+)$ and for every $\alpha \in \kappa$ let I_α be defined as follows:

$$I_\alpha = \bigcup \left\{ X_h : h \in 2^\kappa \setminus \{\overline{1}\} \ \wedge \ h(\alpha) = 0 \right\},$$

where $\overline{1}$ denotes the function $f : \kappa \to 2$ with constant value 1.

Let $\mathcal{I} = \{I_\alpha : \alpha \in \kappa\}$. It is clear that if $h \in 2^\kappa \setminus \{\overline{1}\}$ then $\mathcal{I}^h \supseteq X_h$ and, as $|X_h| = \kappa^+$, we have that $|\mathcal{I}^h| = \kappa^+$, which proves that $\mathcal{I}$ is strongly κ^+-independent. □

The following results are simple corollaries of Theorem 1.8.

Corollary 1.9. *There exists an infinite strongly ω_1-independent family if and only if* CH *is satisfied, thus, the existence of an infinite strongly ω_1-independent family is independent from* ZFC*.*

Corollary 1.10. *Let κ be an inaccessible cardinal (limit and regular) such that for every infinite cardinal $\lambda < \kappa$ it exists a strongly λ^+-independent family of cardinality λ, then κ is strongly inaccessible.*

Proof. We only need to verify that κ is a strong limit cardinal. Let $\lambda \in \kappa$; as κ is limit it follows that $\lambda^+ < \kappa$. On the other hand, since there exists a λ^+-strongly independent family of size λ, by Theorem 1.8, $2^\lambda = \lambda^+$ and so $2^\lambda < \kappa$, which finishes the proof. □

Corollary 1.11. *If κ is inaccessible and for every $\lambda < \kappa$ there is a strongly λ-independent family of cardinality λ, then κ is strongly inaccessible.*

Although we already know some sufficient conditions on κ for the existence of strongly κ-independent families, an interesting property of these families is that the collection of all such families does not satisfy the conditions to apply Zorn's Lemma (unlike classical independent families), which is the standard way to prove the existence of maximal objects with some property.

Definition 1.12. A strongly κ-independent family $\mathcal{I}$ is *maximal* if there is no other strongly κ-independent family that properly extends it.

Kunen proved that if κ is regular, $2^{<\kappa} = \kappa$, and $\mathcal{J}$ is a κ^+-saturated (see below Definition 3.1) κ-complete ideal on κ with $\mathcal{B}(\mathrm{FF}_{<\kappa}(2^\kappa))$ isomorphic to $\mathcal{P}(\kappa)/\mathcal{J}$, then there is a maximal strongly κ-independent family.[4] From this, he deduces that if κ is measurable over M, then there is a forcing extension $M[G]$ such that κ is strongly inaccessible in $M[G]$ (not necessarily measurable), and there is a maximal strongly κ-independent family. Reciprocally, Kunen also showed that if $\kappa > \omega$ is regular and there exists a strongly κ-independent family, then $2^{<\kappa} = \kappa$ and there is a non-trivial κ^+-saturated κ-complete ideal over κ. The existence of such ideal has the consistency strength of the existence of a measurable cardinal, see [10, p. 213]. Another recent result in the same direction is the following:

Theorem 1.13. *[1] Let κ be a supercompact cardinal.*

(1) There is a forcing extension in which for all κ-directed closed posets $\mathbb{P}$ that force $2^\kappa < \kappa^{+\omega}$, they force the existence of strongly κ-independent families.
(2) Suppose GCH *and $\kappa_1 > \kappa$ is measurable. Then there are generic extensions in which there are two maximal strongly κ-independent families of different cardinalities.*

However, it remains open whether it is consistent that there exists a cardinal κ such that the minimal cardinality of a maximal strongly κ-independent family is strictly between κ^+ and 2^κ; this was asked by Eskew and Fischer in [1].

Trivially, every strongly κ-independent family is κ-independent. However, it is straightforward to construct (maximal) κ-independent families that are not strongly κ-independent.

Proposition 1.14. *For every infinite cardinal $\kappa > \omega$ there exists a κ-independent family that is not strongly κ-independent.*

Proof. We know that there exists a bijection between κ and $\omega \times \kappa$, so we are going to construct the desired family on $\kappa \times \omega$. For every $n \in \omega$ let $I_n = \kappa \times C_n$, where the C_n are as in the Example 1.4, and let $\mathcal{I} = \{I_n : n \in \omega\}$.

Clearly if $h; \omega \to 2$ is finite, then for every $\alpha \in \kappa$ we have that $(\{\alpha\} \times \omega) \cap \mathcal{I}^h$ is infinite, in particular $\mathcal{I}^h$ has size κ. On the other hand, if $h : \omega \to 2$ is such that $h^{-1}[\{0\}]$ is infinite, then for every $\alpha \in \kappa$ we have that $(\{\alpha\} \times \omega) \cap \mathcal{I}^h = \emptyset$, which implies that $\mathcal{I}^h = \emptyset$, thus $\mathcal{I}$ is as we wanted. □

It is natural to ask if consistently there is a cardinal κ with nice reflection properties or some sort of compactness principle for which the fact that the finite Boolean combinations are unbounded implies that every Boolean combination of length less than κ is also unbounded, however Proposition 1.14 answers this in the negative since in particular it implies that on every cardinal κ there is a family $\mathcal{I}$ such that every finite Boolean combination is unbounded but plenty of its countable Boolean combinations are empty. Moreover, the family constructed in the proof of Proposition 1.14 can be extended to a maximal κ-independent family $\mathcal{J}$, and since $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{J}$ is not strongly κ-independent either, thus we have the next corollary.

Corollary 1.15. *For every infinite cardinal $\kappa > \omega$ there exists a maximal κ-independent family that is not strongly κ-independent.*

As in the classical case of κ-independent families, a standard diagonalization argument shows that strongly κ-independent families small in cardinality are not maximal, we add here a proof for completeness.

[4] Here $\mathcal{B}(\mathrm{FF}_{<\kappa}(2^\kappa))$ denotes the unique complete Boolean algebra such that $\mathrm{FF}_{<\kappa}(2^\kappa)$ is densely embedded.

Proposition 1.16. *If $\mathcal{I}$ is a strongly κ-independent family such that $|\mathcal{I}| < \kappa$, then there exists a strongly κ-independent family $\mathcal{J}$ such that $\mathcal{I} \subsetneq \mathcal{J}$, i.e., $\mathcal{I}$ is not maximal as a strongly κ-independent family.*

Proof. Let $\mathcal{I} = \{I_\alpha : \alpha \in \lambda\}$ with $\lambda < \kappa$ and for each $h : \lambda \to 2$ let $X_h = \mathcal{I}^h$. Now each set X_h is of cardinality κ and if $h, g \in 2^\lambda$ are different then $X_h \cap X_g = \emptyset$, this implies that $2^\lambda \leq \kappa$. Let $\langle Y_\alpha : \alpha \in \kappa\rangle$ be an enumeration of $\{X_h : h \in 2^\lambda\}$ such that every X_h appears κ times. Let $a_0, b_0 \in Y_0$ be such that $a_0 < b_0$ and suppose that a_β and b_β have been already defined for all $\beta < \alpha$. Since Y_α has cardinality κ there are $a_\alpha, b_\alpha \in Y_\alpha$ such that for all $\beta \in \alpha$ it holds that $a_\beta, b_\beta < a_\alpha$ and also $a_\alpha < b_\alpha$. Now let $Z = \{a_\alpha : \alpha \in \kappa\}$. By the construction of Z we have that $Z \cap X_h$ and $(\kappa \setminus Z) \cap X_h$ have cardinality κ for all $h \in 2^\lambda$, that is, $\mathcal{I} \cup \{Z\}$ is a strongly κ-independent family. □

Note that the above proof is not applicable to strongly κ-independent families of cardinality κ. Besides that, what is established by Proposition 1.16 does not follow from the fact that every κ-independent family $\mathcal{I}$ of size smaller than κ can be properly extended to another κ-independent family $\mathcal{J}$, since, in principle, there is nothing to guarantee that the κ-independent family $\mathcal{J}$ is indeed strongly κ-independent.

The following shows, in the same direction of Proposition 1.16, that another class of strongly independent families are not maximal neither.

Definition 1.17. Let κ be an infinite cardinal.

(1) Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ and $X \subseteq \kappa$, we say that X *splits* $\mathcal{F}$ if $Y \cap X$ and $Y \setminus X$ have size κ for all $Y \in \mathcal{F}$.
(2) A family $\mathcal{R} \subseteq \mathcal{P}(\kappa)$ is *unsplittable* (or *reaping*) if there is not $X \subseteq \kappa$ that splits $\mathcal{R}$.
(3) $\mathfrak{r}(\kappa)$ is the smallest cardinality of a unsplittable family on κ.

Theorem 1.18. *[3] Let κ be an infinite regular cardinal. If $\mathcal{I}$ is a strongly κ-independent family such that $|\{\mathcal{I}^h : h; \mathcal{I} \to 2 \wedge |h| < \kappa\}| < \mathfrak{r}(\kappa)$ then $\mathcal{I}$ is not maximal.*

In the next section we take a different approach to generalized the classical case. Again the property of being maximal for those is perhaps even harder. For example, we were unable to prove that a countable $\mathcal{C}$-independent family cannot be maximal. See Theorem 2.10.

2. $\mathcal{F}$-independent families

Let $\mathcal{F}$ be a filter on κ. A subset $X \subseteq \kappa$ is $\mathcal{F}$-positive if $X \cap Y \neq \emptyset$ for every $Y \in \mathcal{F}$; we denote the family of $\mathcal{F}$-positive subsets by $\mathcal{F}^+$. If $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ is an ideal then $\mathcal{J}^+ = \{X \subseteq \kappa : X \notin \mathcal{J}\}$.

If $\mathcal{F}$ a filter on a cardinal κ, we denote by $\mathcal{F}^*$ its dual ideal, i.e., the ideal $\{X \subseteq \kappa : \kappa \setminus X \in \mathcal{F}\}$.

Definition 2.1. A family $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is $\mathcal{F}$-*independent* if every finite Boolean combination of $\mathcal{I}$ is in $\mathcal{F}^+$. Similarly if $\mathcal{J}$ is an ideal then $\mathcal{I}$ is $\mathcal{J}$-*independent* if every finite Boolean combination of $\mathcal{I}$ is in $\mathcal{J}^+$.

Note that once we fix a filter $\mathcal{F}$, we know the cardinal κ it is on since $\bigcup \mathcal{F} = \kappa$; in this way, the most natural is to define a $\mathcal{F}$-independent family as a subfamily of $\mathcal{P}(\kappa)$. However, the case of ideals is a little more subtle since an ideal itself *does not remember* the cardinal it is on. For example, if $\mathcal{J} \subseteq \mathcal{P}(\omega)$ is an ideal, then in particular $\mathcal{J} \subseteq \mathcal{P}(\omega_1)$, so in this case when talking about a $\mathcal{J}$-independent family, there is certain ambiguity regarding which cardinal the family should be on and in particular how the Boolean combinations are taken. To avoid this ambiguity, and as we did in Section 1, when we talk about a $\mathcal{J}$-independent family $\mathcal{I}$ and indicate that $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ we will implicitly assume that the Boolean combinations are taken in κ.

A similar issue arises when $\mathcal{J}$ is an ideal and we want to talk about its dual filter as we need to know the cardinal κ on which the complements of elements of $\mathcal{J}$ are taken. To solve this, when we indicate that $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ is an ideal and we talk about its dual filter, we will assume that this filter is defined on κ.

Note that a family is $\mathcal{F}$-independent if and only if it is $\mathcal{F}^*$-independent. On the other hand, if $\mathcal{F}_r$ is the Fréchet filter (on ω), then a family is $\mathcal{F}_r$-independent if and only if it is ω-independent. It is also clear that if $\mathcal{I}$ is a $\mathcal{F}$-independent family and $X \in \mathcal{I}$, then X is $\mathcal{F}$-*double positive*, that is, $X, \kappa \setminus X \in \mathcal{F}^+$, consequently if $\mathcal{F}$ is an ultrafilter, there are no $\mathcal{F}$-independent families. The natural question is to know for which filters (or ideals) $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ (in addition to the Fréchet's one) there is a $\mathcal{F}$-independent family.

Proposition 2.2. *Let $\mathcal{F}$ be a principal filter, i.e., $\mathcal{F} = \{A \subseteq \kappa : B \subseteq A\}$ for some $B \subseteq \kappa$. Then:*

(1) If B is finite then there are not $\mathcal{F}$-independent infinite families. Furthermore, if $|B| = n$, there are not $\mathcal{F}$-independent families of cardinality n.
(2) If $|B| = \lambda \geq \omega$, then there exists an $\mathcal{F}$-independent family $\mathcal{I}$ of cardinality 2^λ. On the other hand, if $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{J}| \geq (2^\lambda)^+$, then $\mathcal{J}$ is not $\mathcal{F}$-independent.

Proof. (1) Note that $\mathcal{F}^+ = \{X \subseteq \kappa : X \cap B \neq \emptyset\}$. Let $B = \{x_0, \dots, x_{n-1}\}$ and suppose that $X_0, \dots, X_{n-1} \in \mathcal{I}$ are all distinct, where $\mathcal{I}$ is an $\mathcal{F}$-independent family. For each $i \in n$, if $x_i \in X_i$ let $h(i) = 1$ and $h(i) = 0$ otherwise; so we have that $x_i \notin X_i^{h(i)}$. Then for every $x \in B$ we have that:

$$x \notin \bigcap_{i \in n} X_i^{h(i)} = \mathcal{I}^h,$$

so $\mathcal{I}^h \cap B = \emptyset$ and therefore $\mathcal{I}^h \notin \mathcal{F}^+$, which contradicts the fact that $\mathcal{I}$ is $\mathcal{F}$-independent.

(2) Again note that $\mathcal{F}^+ = \{X \subseteq \kappa : X \cap B \neq \emptyset\}$. Now let $\mathcal{I} = \{X_\alpha : \alpha \in 2^\lambda\}$ be an independent family of subsets of B and for each $\alpha \in 2^\lambda$ let $Y_\alpha = X_\alpha \cup (\kappa \setminus B)$ and let $\widehat{\mathcal{I}} = \{Y_\alpha : \alpha \in 2^\lambda\}$. Clearly if $h; 2^\lambda \to 2$ is finite then $\mathcal{I}^h \subseteq \widehat{\mathcal{I}}^h$ and as $\mathcal{I}$ is independent on B we have that:

$$\emptyset \neq B \cap \mathcal{I}^h = B \cap \widehat{\mathcal{I}}^h,$$

which proves that $\widehat{\mathcal{I}}^h \in \mathcal{F}^+$, therefore $\widehat{\mathcal{I}}$ is $\mathcal{F}$-independent.

If $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ has cardinality at least $(2^\lambda)^+$, as $|B| = \lambda$, there exist $X, Y \in \mathcal{J}$ distinct such that $X \cap B = Y \cap B$, but then $(X \setminus Y) \cap B = \emptyset$, which proves that $X \setminus Y \notin \mathcal{F}^+$, thus $\mathcal{J}$ is not $\mathcal{F}$-independent. □

As anticipated, the two generalizations of independence studied in this work are compatible with each other, that is, we can *merge* the two notions in order to obtain families with more combinatorial properties.

Definition 2.3. Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a filter (respectively $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ an ideal). A family $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is *strongly $\mathcal{F}$-independent* (respectively *strongly $\mathcal{J}$-independent*) if every Boolean combination of length less than κ of $\mathcal{I}$ is in $\mathcal{F}^+$ (respectively in $\mathcal{J}^+$).

We will study a little more of these families below.

2.1. $\mathcal{C}$-independent families

For each regular cardinal κ let $\mathcal{C}_\kappa \subseteq \mathcal{P}(\kappa)$ be the club filter, that is, the filter generated by closed and unbounded sets (when the context is clear we will call $\mathcal{C}_\kappa$ simply as $\mathcal{C}$). $\mathcal{C}_{\omega_1}$ is a very important filter in the study of the combinatorics of ω_1, therefore a couple of questions arise naturally: Are there $\mathcal{C}_{\omega_1}$-independent families? Is every maximal $\mathcal{C}$-independent family strongly $\mathcal{C}$-independent? Answers to these questions can be found in Proposition 2.6 and Corollary 2.7, respectively.

First of all, let us note that as for every filter $\mathcal{F}$, the union of a chain of $\mathcal{F}$-independent families is again an $\mathcal{F}$-independent family. Therefore, if there exist $\mathcal{F}$-independent families, then there are maximal ones (by Zorn's Lemma).

Remember that $\mathcal{C}$-positive sets are called *stationary* sets; one of the most important results about stationary sets is the following:

Lemma 2.4 *([15], [10]). For each uncountable regular cardinal κ we have that κ is the union of as many as κ disjoint stationary sets.*

Corollary 2.5. *For each uncountable regular cardinal κ and each $\lambda \leq \kappa$ we have that κ is the union of λ disjoint stationary sets.*

The following two results are consequences of this last corollary; their proofs follow the scheme of the proof of Proposition 1.14.

Proposition 2.6. *For every uncountable regular cardinal κ there exists a countable $\mathcal{C}$-independent family.*

Proof. By Corollary 2.5 there is a countable collection $\{X_s : s \in 2^{<\omega}\}$ of disjoint stationary subsets whose union is κ, say indexed by the set $2^{<\omega}$.

Now, for every $n \in \omega$, let $I_n \subseteq \kappa$ be defined as follows:

$$I_n = \bigcup\{X_s : s \in 2^{<\omega} \wedge n \in dom(s) \wedge s(n) = 0\}.$$

It turns out that $\mathcal{I} = \{I_n : n \in \omega\}$ is a $\mathcal{C}$-independent family, since every finite Boolean combination of $\mathcal{I}$ contains some combination of the form

$$\bigcap\{I_n^{s(n)} : n \in dom(s)\}$$

for some $s \in 2^{<\omega}$ and also:

$$X_s \subseteq \bigcap\{I_n^{s(n)} : n \in dom(s)\},$$

which proves that every finite Boolean combination of $\mathcal{I}$ contains a stationary set, therefore is stationary. □

Corollary 2.7. *For any cardinal $\kappa \geq \omega_1$ it exists a $\mathcal{C}$-independent maximal family on κ that is not strongly $\mathcal{C}$-independent.*

Proof. Let $\{X_m : m \in \omega\}$ be a partition of κ into stationary sets. Now for every $n \in \omega$ let $Y_n = \bigcup\{X_m : m \in C_n\}$, where the C_n are as in the Example 1.4. Consider $\mathcal{I} = \{Y_n : n \in \omega\}$; then it is easy that $\mathcal{I}$ is $\mathcal{C}$-independent but for $h; \omega \to 2$ such that $h^{-1}[\{0\}]$ is infinite we have that $\mathcal{I}^h = \emptyset$, which proves that $\mathcal{I}$ is not strongly $\mathcal{C}$-independent. Extending $\mathcal{I}$ to a maximal $\mathcal{C}$-independent family the result is obtained.[5] □

Theorem 2.8. *The following statements are equivalent for a cardinal κ:*

(1) $2^\kappa = \kappa^+$.

[5] As mentioned earlier, this can be accomplished by applying Zorn's Lemma.

(2) There exists a strongly independent family on κ^+ of size κ.
(3) There exists a strongly $\mathcal{C}$-independent family on κ^+ of size at least κ.

Proof. We only prove *(1)⇒(3)*. Let $\{X_f : f \in 2^\kappa\}$ be a partition of κ^+ into stationary sets and for every $\alpha \in \kappa$ let I_α be defined by

$$I_\alpha = \bigcup \{X_f : f \in 2^\omega \ \wedge \ (f(\alpha) = 0)\}.$$

Let $\mathcal{I} = \{I_\alpha : \alpha \in \kappa\}$. It is clear that if $f; \kappa \to 2$ then $\mathcal{I}^f \supseteq X_h$ for some $h \in 2^\kappa$ and, as X_h is stationary, $\mathcal{I}^f$ is stationary too, which proves that $\mathcal{I}$ is strongly $\mathcal{C}$-independent. □

We now know that there are countable $\mathcal{C}$-independent families on ω_1. Are there uncountable $\mathcal{C}$-independent families on ω_1? Furthermore, are there $\mathcal{C}$-independent families of cardinality 2^{ω_1}? This is answered positively in the following.

Theorem 2.9. *Let κ and λ be cardinals such that $\omega \leq \lambda \leq 2^\kappa$ and κ is regular. Then, on κ, there is a $\mathcal{C}$-independent family of cardinality λ.*

Proof. Let $\{X_\beta : \beta \in \kappa\}$ be a partition of κ into stationary sets. Now let $\mathcal{I} = \{I_\alpha : \alpha \in \lambda\}$ be an independent family of cardinality λ on κ. For every $\alpha \in \lambda$, let $\widehat{I_\alpha} \subseteq \kappa$ be defined as follows:

$$\widehat{I_\alpha} = \bigcup \{X_\beta : \beta \in I_\alpha\}.$$

Now let $\widehat{\mathcal{I}} = \{\widehat{I_\alpha} : \alpha \in \lambda\}$. Clearly $\widehat{\mathcal{I}}$ has size λ, then the only thing left to prove is that it is a $\mathcal{C}$-independent family. Let $s; \lambda \to 2$ be finite, we want to see that $\widehat{\mathcal{I}}^s$ is stationary. Since $\mathcal{I}$ is independent there is $\beta \in \mathcal{I}^s$, but this means that if $s(\alpha) = 0$ then $X_\beta \subseteq \widehat{I_\alpha}$ and if $s(\alpha) = 1$ then $X_\beta \cap \widehat{I_\alpha} = \emptyset$, that is, $X_\beta \subseteq \widehat{\mathcal{I}}^s$, and since X_β is stationary $\widehat{\mathcal{I}}^s$ is also stationary. □

As in the classical case of independent families, one would expect that the countable $\mathcal{C}_{\omega_1}$-independent families are not maximal; however, it seems complicated to establish that. Our ideas about generalizing the classical proof, doing a disjoint refinement of the envelope or using a $\diamondsuit^\sharp$-sequence have failed. The following is a modification of the main construction from [7].

Theorem 2.10. *If $\mathbf{V} = \mathbf{L}$, then every countable $\mathcal{C}_{\omega_1}$-independent family can not be maximal.*

Proof. Let $\mathcal{I}$ be a countable $\mathcal{C}$-independent family, and let $\{E_n : n \in \omega\}$ be an enumeration of its ω-envelope. For each limit ordinal $\gamma < \omega_1$ set

$$A_\gamma = \{\alpha < \omega_1 : L(\alpha) \models \mathsf{ZF}^- \ \wedge \ \gamma = \omega_1^{L(\alpha)}\}.$$

Since $\{\varrho < \omega_1 : L(\varrho) \prec L(\omega_1)\}$ is unbounded in ω_1, it follows that A_γ is at most countable for each limit $\gamma < \omega_1$. It is also known that $\{\gamma < \omega_1 : A_\gamma \neq \emptyset\}$ contains a club. Let

$$\mathcal{G}_\gamma = \{C \subseteq \gamma : C \text{ is club in } \gamma \ \wedge \ (\exists \alpha \in A_\gamma)(C \in L(\alpha))\}.$$

Then $\mathcal{G}_\gamma$ is countable and since ZF^- suffices to prove that the intersection of a finite collection of club subsets is a club subset, it follows that $\mathcal{G}_\gamma$ is closed under finite intersections.

Consider as well

$$\mathcal{S}_\gamma = \{S \subseteq \gamma : (\exists \alpha \in A_\gamma)(S \in L(\alpha)) \ \wedge \ (\forall C \in \mathcal{G}_\gamma)(C \cap S \neq \emptyset)\}.$$

Once again $\mathcal{S}_\gamma$ is countable; fix an enumeration $\{S_n : n \in \omega\}$ of $\mathcal{S}_\gamma$ in which each element appears infinitely often and some simple enumeration $\{C_n : n \in \omega\}$ of $\mathcal{G}_\gamma$. Now consider a cofinal sequence $\langle \alpha_n : n \in \omega \rangle$ in A_γ such that

$$S_n \in L(\alpha_n) \ \wedge \ (\forall m \le n)(C_m \in L(\alpha_n)).$$

Since $L(\alpha_0) \models$ "S_0 is stationary in γ" pick

$$\xi_0 \in S_0 \cap C_0 \quad \text{and} \quad \eta_0 \in S_0 \cap C_0 \setminus (\xi_0 + 1),$$

and recursively

$$\xi_{n+1} \in (S_{n+1} \cap \bigcap_{k \le n+1} C_k) \setminus (\eta_n + 1) \quad \text{and} \quad \eta_{n+1} \in (S_{n+1} \cap \bigcap_{k \le n} C_k) \setminus (\xi_{n+1} + 1),$$

for all $n \in \omega$. This way we have built two disjoint subsets $G_\gamma = \{\xi_n : n \in \omega\}$ and $H_\gamma = \{\eta_n : n \in \omega\}$.

Put $G = \bigcup\{G_\gamma : \gamma \in \mathrm{Lim}(\omega_1)\}$ and $H = \bigcup\{H_\gamma : \gamma \in \mathrm{Lim}(\omega_1)\}$.

Claim: $(\forall k \in \omega)(E_k \cap G$ is stationary$)$.

Fix a club subset $C \subseteq \omega_1$. Define recursively a sequence of elementary submodels $M_\nu \prec L(\omega_2)$ for $\nu < \omega_2$ as follows:

- M_0 is the smallest $M \prec L(\omega_2)$ such that $\{E_n : n \in \omega\}, C \in M$,
- $M_{\nu+1}$ is the smallest $M \prec L(\omega_2)$ such that $M_\nu \cup \{M_\nu\} \subseteq M$,
- $M_\xi = \bigcup_{\nu<\xi} M_\nu$ whenever ξ is a limit ordinal.

By the Condensation Lemma, $M_\nu \cap L(\omega_1)$ is transitive, set $\alpha_\nu = M_\nu \cap \omega_1$. Then $\langle \alpha_\nu : \nu < \omega_1 \rangle$ is a normal sequence in ω_1. Use Mostowski's Collapse $\pi_\nu : M_\nu \cong L(\beta_\nu)$ to get

- $\pi_\nu \restriction L(\alpha_\nu) = \mathrm{id} \restriction L(\alpha_\nu)$,
- $\pi_\nu(\omega_1) = \alpha_\nu$,
- $\pi_\nu(C) = C \cap \alpha_\nu$,
- $(\forall n \in \omega)(\pi_\nu(E_n) = E_n \cap \alpha_\nu)$.

Consider the set K of limit points of $\langle \alpha_\nu : \nu < \omega_1 \rangle$. Obviously K is a club in ω_1, if $\gamma \in K$, then

$$\gamma = \sup_{\nu<\zeta} \alpha_\nu = \sup_{\nu<\zeta} \beta_\nu,$$

for some ordinal $\zeta < \omega_1$, and hence $\gamma = \alpha_\zeta$. To see this, it is enough to show $\alpha_\nu < \beta_\nu < \alpha_{\nu+1}$. Clearly $\alpha_\nu < \beta_\nu$. Since β_ν is definable from M_ν as $L(\beta_\nu)$ is the transitive collapse of M_ν and that definition relativizes to $L(\omega_2)$. Thus $\beta_\nu \in M_{\nu+1}$ as $M_\nu \in M_{\nu+1} \prec L(\omega_2)$. Henceforth $\beta_\nu \in \alpha_{\nu+1}$.

Note that $\beta_\zeta \in A_\gamma$ since $L(\beta_\zeta) \models \gamma = \omega_1$ and $L(\beta_\zeta) \models \mathsf{ZF}^-$. Thus $C \cap \gamma = \pi_\zeta(C) \in L(\beta_\zeta)$ and of course $L(\beta_\zeta)$ models that $\pi_\zeta(C)$ is a club in γ. This implies $C \cap \gamma \in \mathcal{G}_\gamma$. It is also true that $E_k \cap \gamma \in L(\beta_\zeta)$, then $E_k \cap \gamma = S_n \in \mathcal{S}_\gamma$, for infinitely many $n \in \omega$. Since $G_\gamma \setminus (C \cap \gamma)$ is finite and G_γ is built in such a way that $G_\gamma \cap (E_k \cap \gamma)$ is infinite, this shows that $E_k \cap G$ is stationary in ω_1.

Analogously $E_k \cap H$ is stationary in ω_1 for all $k \in \omega$. It follows that $\mathcal{I} \cup \{A\}$ is also $\mathcal{C}$-independent. □

Observe, in the last proof, that it is easily possible that $G \cap H \neq \emptyset$; however, it is not hard to show that $G \setminus H$ and $H \setminus G$ are stationary as well.

Maximal $\mathcal{C}$-independent families have many properties analogous to those of maximal independent ones in the classical case. For example, it is easy to prove that if $\mathcal{I}$ is $\mathcal{C}$-independent and finite then it is not maximal. Indeed, let us say that $\mathcal{I} = \{I_i : i \in n\}$ for some $n \in \omega$ and note that for each $s : n \to 2$, the set $\mathcal{I}^s = \bigcap_{i \in n} I^{s(i)}$ is stationary; furthermore, if $s, t : n \to 2$ are different, $\mathcal{I}^s$ and $\mathcal{I}^t$ are disjoint. For each $s : n \to 2$ let A_s and B_s be a partition of $\mathcal{I}^s$ into two disjoint stationary sets and let $A = \bigcup\{A_s : s \in 2^n\}$. It is clear that $A \notin \mathcal{I}$ and $\mathcal{I} \cup \{A\}$ is $\mathcal{C}$-independent.

Note that, since we can always split a stationary set into two stationary subsets, the above guarantees that we can recursively construct $\mathcal{C}$-independent families of cardinality $n \in \omega$ and thus obtain a countable $\mathcal{C}$-independent family. The advantage of this method is that it only requires the fact that a stationary set can be split into two stationary sets and not into infinite ones.

It must be clear that our method from the last paragraph is too far from working in the infinite case. Although $\mathbf{V} = \mathbf{L}$ is a reasonable hypothesis, we conjecture that the assertion in our last theorem may be establish without further hypothesis beyond the usual. A model where there is a countable maximal $\mathcal{C}_{\omega_1}$-independent would be a very interesting one.

Question 2.11. Is it true in ZFC that every countable $\mathcal{C}_{\omega_1}$-independent family is not maximal?

In analogy to the classical case, we may introduce $\mathfrak{i}_{\mathcal{C}_{\omega_1}}$ as the minimum size of a maximal $\mathcal{C}_{\omega_1}$-independent family. So with this terminology the former question becomes: Is it true in ZFC that $\mathfrak{i}_{\mathcal{C}_{\omega_1}} \geq \omega_1$?

2.1.1. Dense $\mathcal{C}$-independent families

All the properties shown next for $\mathcal{C}$-independent families were proved for the classic independence by Goldstern and Shelah in [5], this proves that $\mathcal{C}$-independent families on ω_1 behave similarly as the ω-independent ones.

Definition 2.12. If $\mathcal{I}$ is a $\mathcal{C}$-independent family then we define the *ideal associated* to $\mathcal{I}$ as:

$$\mathcal{J}_{\mathcal{I}} = \{A \subseteq \omega_1 : (\forall f \in \mathrm{FF}_{<\omega}(\mathcal{I}))(\exists g \in \mathrm{FF}_{<\omega}(\mathcal{I}))(g \supseteq f \wedge \mathcal{I}^g \cap A \text{ is not stationary})\}.$$

Clearly $\mathcal{J}_{\mathcal{I}}$ is an ideal that contains the ideal of the non-stationary sets.

Definition 2.13.

(1) If $X, Y \subseteq \omega_1$, we say that X is NS-*almost contained* in Y if $X \setminus Y$ is not a stationary set and we denote this by $X \subseteq_{\mathrm{NS}} Y$.
(2) For a family $\mathcal{X}$ of subsets of ω_1 and $Y \subseteq \omega_1$, we say that Y is NS-*pseudointersection* of $\mathcal{X}$ if $Y \subseteq_{\mathrm{NS}} X$ for every $X \in \mathcal{X}$.

In this definition we focus in the ideal of non-stationary sets in ω_1; however, it is straightforward defining the relation $X \subseteq_{\mathcal{J}} Y$, for any other ideal $\mathcal{J}$.

Definition 2.14. A $\mathcal{C}$-independent maximal family is *dense* if for every $A \in {\mathcal{J}_{\mathcal{I}}}^+$ it exists $g \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that $\mathcal{I}^g \subseteq_{\mathrm{NS}} A$.

This can be interpreted as follows: a $\mathcal{C}$-independent family is dense if the envelope of $\mathcal{I}$ is a *base* of ${\mathcal{J}_{\mathcal{I}}}^+$; let us also note that for all $f \in \mathrm{FF}_{<\omega}(\mathcal{I})$ we have that $\mathcal{I}^f \in {\mathcal{J}_{\mathcal{I}}}^+$, since f itself is a witness of this.

Next we use the following standard notation, if $\mathcal{A} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, then $\mathcal{A} \restriction Y$ is the family $\{A \cap Y : A \in \mathcal{A}\}$.

Proposition 2.15. *If $\mathcal{I}$ is a maximal $\mathcal{C}$-independent family, there exists $f \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that for every $g \in \mathrm{FF}_{<\omega}(\mathcal{I})$ with $g \supseteq f$, $\mathcal{I} \upharpoonright \mathcal{I}^g$ is maximal.*

Proof. Let $\{f_n : n \in \omega\}$ be a maximal family with the following properties:

(1) If $n \neq m$, f_n and f_m are incompatible.
(2) $\mathcal{I} \upharpoonright \mathcal{I}^{f_n}$ is not maximal for every $n \in \omega$.

Note that by condition (1) and since $\mathrm{FF}_{<\omega}(\mathcal{I})$ is ccc, this collection is at most countable (in principle it could be finite but assume without loss of generality that it is countable).

Now, for every $n \in \omega$ let $A_n \subseteq \mathcal{I}^{f_n}$ be such that $\mathcal{I} \upharpoonright \mathcal{I}^{f_n} \cup \{A_n\}$ is $\mathcal{C}$-independent on $\mathcal{I}^{f_n}$ and let $A = \bigcup_{n\in\omega} A_n$. Since $\mathcal{I}$ is maximal it exists $f \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that $\mathcal{I}^f \cap A$ or $\mathcal{I}^f \setminus A$ is not stationary. Let us suppose without loss of generality that $\mathcal{I}^f \cap A$ is not stationary. We claim that f is incompatible with every f_n; to see this, suppose that f and f_n are compatible, i.e., suppose that $f \cup f_n$ is a function. Thus $\mathcal{I}^{f\cup f_n} \in \mathrm{ENV}_{<\omega}(\mathcal{I} \upharpoonright \mathcal{I}^{f_n})$, in particular we have that:

$$\mathcal{I}^f \cap A \supseteq \mathcal{I}^{f\cup f_n} \cap A \supseteq \mathcal{I}^{f\cup f_n} \cap A_n,$$

but this is impossible, since in that case $\mathcal{I}^{f\cup f_n} \cap A_n$ is stationary as $\mathcal{I}^f \cap A$ is not.

Since f is incompatible with every f_n then so is every $g \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that $g \subseteq f$, therefore $\mathcal{I} \upharpoonright \mathcal{I}^g$ is maximal, otherwise the maximality of $\{f_n : n \in \omega\}$ would be contradicted. $\square$

Lemma 2.16. *If $\mathcal{I}$ is a $\mathcal{C}$-independent maximal family such that for every $f \in \mathrm{FF}_{<\omega}(\mathcal{I})$ the family $\mathcal{I} \upharpoonright \mathcal{I}^f$ is maximal, then $\mathcal{I}$ is dense.*

Proof. Let $A \in \mathcal{J}_{\mathcal{I}}{}^+$, this means that there exists $f \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that for every $g \in \mathrm{FF}_{<\omega}(\mathcal{I})$ that extends to f we have that $\mathcal{I}^g \cap A$ is stationary. As $\mathcal{I} \upharpoonright \mathcal{I}^f$ is maximal, it exists $g \in \mathrm{FF}_{<\omega}(\mathcal{I})$, $g \supseteq f$ such that either $\mathcal{I}^g \cap A$ or $\mathcal{I}^g \setminus A$ is not stationary, but we know that $\mathcal{I}^g \cap A$ is stationary, then necessarily $\mathcal{I}^g \setminus A$ is not, i.e., $\mathcal{I}^g \subseteq_{\mathrm{NS}} A$, which is what we wanted. $\square$

Proposition 2.17. *If $\mathcal{I}$ is a maximal $\mathcal{C}$-independent family which is dense, then ${}^{\mathcal{P}(\omega_1)}/_{\mathcal{J}_{\mathcal{I}}}$ is ccc.*

Proof. By contradiction. Suppose that $\{X_\alpha : \alpha \in \omega_1\} \subseteq \mathcal{J}_{\mathcal{I}}{}^+$ is such that if $\alpha \neq \beta$ then $X_\alpha \cap X_\beta \in \mathcal{J}_{\mathcal{I}}$. Since $\mathcal{I}$ is a dense family, for every $\alpha \in \omega_1$ it exists $f_\alpha \in \mathrm{FF}_{<\omega}(\mathcal{I})$ such that $\mathcal{I}^{f_\alpha} \subseteq_{\mathrm{NS}} X_\alpha$. Now if $\alpha \neq \beta$ then f_α and f_β are incompatible, otherwise $\mathcal{I}^{f_\alpha\cup f_\beta} = \mathcal{I}^{f_\alpha} \cap \mathcal{I}^{f_\beta} \subseteq_{\mathrm{NS}} X_\alpha \cap X_\beta \in \mathcal{J}_{\mathcal{I}}$. But now $\mathcal{I}^{f_\alpha\cup f_\beta} \in \mathcal{J}_{\mathcal{I}}$ (as $\mathcal{J}_{\mathcal{I}}$ contains the non-stationary sets), and this is a contradiction as $\mathrm{ENV}_{<\omega}(\mathcal{I}) \subseteq \mathcal{J}_{\mathcal{I}}{}^+$.

Thus the family $\{f_\alpha : \alpha \in \omega_1\}$ is an antichain in $\mathrm{FF}_{<\omega}(\mathcal{I})$, but this contradicts the fact that $\mathrm{FF}_{<\omega}(\mathcal{I})$ is ccc. $\square$

Proposition 2.17 appears in [5] for the case of classical independent families. There it is employed as a small part in the proof of the consistency of $\mathfrak{s} = \mathfrak{d} = \mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{c}$. This raises a natural question: can that entire proof, or certain parts of it, be naturally adapted for the invariants that correspond to subfamilies of $\mathcal{P}(\omega_1)$ taking modulo non-stationary? In particular, if we let $\mathfrak{r}_{\mathcal{C}_{\omega_1}} := \min\{|\mathcal{R}| \mid \mathcal{R} \subseteq \mathcal{P}(\omega_1)(\mathcal{R} \text{ is } \mathcal{C}_{\omega_1}\text{-reaping})\}$ where a family $\mathcal{R} \subseteq \mathcal{P}(\omega_1)$ is considered $\mathcal{C}_{\omega_1}$*-reaping* if for all stationary $X \subseteq \omega_1$ there is $R \in \mathcal{R}$ such that $R \subseteq_{\mathrm{NS}} X$ or $R \cap X =_{\mathrm{NS}} \emptyset$, then it is easy to see that $\mathfrak{r}_{\mathcal{C}_{\omega_1}} \leq \mathfrak{i}_{\mathcal{C}_{\omega_1}}$. This way it is very natural to ask:

Question 2.18. It is consistent that $\mathfrak{r}_{\mathcal{C}_{\omega_1}} = \aleph_2 < \aleph_3 = \mathfrak{i}_{\mathcal{C}_{\omega_1}} = 2^{\omega_1}$?

2.1.2. Strongly $\mathcal{C}$-independent families

Lemma 2.19. *Let $\mathcal{E} = \{E_n : n \in \omega\}$ be a nested collection of stationary sets, i.e., $E_{n+1} \subseteq E_n$ for all $n \in \omega$. The following conditions are equivalent:*

(1) $\mathcal{E}$ admits a stationary NS-pseudointersection, that is, there is a stationary set X such that, $X \setminus E_n$ is not stationary, for all $n \in \omega$.
(2) $\bigcap_{n\in\omega} E_n$ is stationary.

Proof. (1) ⇒ (2) Note that

$$X = (X \cap \bigcap_{n\in\omega} E_n) \cup (X \cap (\omega_1 \setminus \bigcap_{n\in\omega} E_n))$$

and one of the two sets forming the union must be stationary. On the other hand:

$$X \cap (\omega_1 \setminus \bigcap_{n\in\omega} E_n) = X \cap (\bigcup_{n\in\omega} \omega_1 \setminus E_n) = \bigcup_{n\in\omega} X \cap (\omega_1 \setminus E_n) = \bigcup_{n\in\omega} X \setminus E_n,$$

and as every $X \setminus E_n$ is not stationary, then neither is $\bigcup_{n\in\omega} X \setminus E_n$, i.e., $X \cap (\omega_1 \setminus \bigcap_{n\in\omega} E_n)$ is not stationary. Necessarily $X \cap \bigcap_{n\in\omega} E_n$ is stationary and consequently $\bigcap_{n\in\omega} E_n$ also is stationary.

(2) ⇒ (1) In this case it is enough to take $X = \bigcap_{n\in\omega} E_n$. □

Corollary 2.20. *Let $\mathcal{I} \subseteq \mathcal{P}(\omega_1)$ be a $\mathcal{C}$-independent family. The following conditions are equivalent:*

(1) $\mathcal{I}$ is strongly $\mathcal{C}$-independent.
(2) For every $f;\mathcal{I} \to 2$ with f countable, the collection $\{\mathcal{I}^{f\restriction n} : n \in \omega\}$ admits a stationary NS-pseudointersection.[6]

Proposition 2.21. *Let $\mathcal{I}$ a countable strongly $\mathcal{C}$-independent family. Then there is $\mathcal{J} \subseteq \mathcal{P}(\omega_1)$ such that $\mathcal{I} \subsetneq \mathcal{J}$ and $\mathcal{J}$ is strongly $\mathcal{C}$-independent, i.e., $\mathcal{I}$ is not a maximal strongly $\mathcal{C}$-independent family.*

Proof. Let $\mathcal{I} = \{I_n : n \in \omega\}$ be a strongly $\mathcal{C}$-independent family. For each $f \in 2^\omega$ consider $X_f = \mathcal{I}^f$. If $f \neq g$ then $X_f \cap X_g = \emptyset$. Now let $\{A_f, B_f\}$ be a partition of X_f into stationary sets and define A by:

$$A = \bigcup_{f\in 2^\omega} A_f.$$

Let us see that $\mathcal{J} =: \mathcal{I} \cup \{A\}$ is strongly $\mathcal{C}$-independent. For this it is enough to see that for all $f \in 2^\omega$, the sets $X_f \cap A$ and $X_f \setminus A$ are both stationary, however $X_f \cap A = A_f$ and $X_f \setminus A = B_f$ are stationary sets. □

Note that, as every strongly $\mathcal{C}$-independent family is $\mathcal{C}$-independent, the family $\mathcal{J}$ constructed in the previous proof is $\mathcal{C}$-independent, then, we get the following.

Corollary 2.22. *Let $\mathcal{I}$ a countable strongly $\mathcal{C}$-independent family (in particular $\mathcal{I}$ is $\mathcal{C}$-independent). Then there is $\mathcal{J} \subseteq \mathcal{P}(\omega_1)$ such that $\mathcal{I} \subsetneq \mathcal{J}$ and $\mathcal{J}$ is $\mathcal{C}$-independent.*

[6] For $f \restriction n$ to make sense, it is enough to enumerate the domain of f and so f can be interpreted as a function in 2^ω.

Note that Corollary 2.22 is a partial answer to Question 2.11, since it implies that if $\mathcal{I}$ is a countable maximal $\mathcal{C}$-independent family, then $\mathcal{I}$ cannot be strongly $\mathcal{C}$-independent.

As we have seen, under CH there are countable $\mathcal{C}$-independent families that are strongly $\mathcal{C}$-independent, on the other hand (without extra hypothesis further than ZFC) there are also countable $\mathcal{C}$-independent families that are *very far* from being strong. This means that there exists $\mathcal{I} = \{I_n : n \in \omega\} \subseteq \mathcal{P}(\omega_1)$ which is $\mathcal{C}$-independent but that for every $h \in 2^\omega$ such that $|h^{-1}(\{0\})| = \omega$ we have that $\mathcal{I}^h = \emptyset$; for example, to construct one of these families it is enough to take $\{X_n : n \in \omega\}$ a partition of ω_1 into stationary sets and define I_n as:

$$I_n = \bigcup_{m \in C_n} X_m,$$

where the C_n are as in Example 1.4, in this way, the family $\{I_n : n \in \omega\}$ fulfills this property.

3. Saturated ideals and $\mathcal{J}$-independent families

Saturation of ideals has been closely related to the study of large cardinals, therefore it constitutes, as we will see in this section, a bridge between these cardinals and the existence of $\mathcal{J}$-independent families on them.

Definition 3.1. Let $\mathcal{J}$ be an ideal on a cardinal κ. Then:

(1) $\mathcal{J}$ is λ-*saturated* if for every collection $\{X_\alpha : \alpha \in \lambda\} \subseteq \mathcal{J}^+$ there exist $\beta < \gamma < \lambda$ such that $X_\beta \cap X_\gamma \in \mathcal{J}^+$.
(2) $\mathrm{sat}(\mathcal{J})$ is the smallest λ such that $\mathcal{J}$ is λ-saturated.

Lemma 3.2. *Let $\mathcal{J}$ be an ideal on a cardinal κ such that $\mathrm{sat}(\mathcal{J}) > \lambda$ for some cardinal λ. Then there exists a $\mathcal{J}$-independent family on κ of cardinality 2^λ.*

Proof. Since $\mathcal{J}$ is not λ-saturated, it exists a collection $\{X_\beta : \beta \in \lambda\} \subseteq \mathcal{J}^+$ such that $X_\beta \cap X_\gamma \in \mathcal{J}$, if $\beta \neq \gamma$. Let $\mathcal{I} = \{I_\alpha : \alpha \in 2^\lambda\}$ be a λ-independent family of cardinality 2^λ.[7] For each $\alpha \in 2^\lambda$, let $\widehat{I_\alpha} \subseteq \kappa$ be defined as follows:

$$\widehat{I_\alpha} = \bigcup \{X_\beta : \beta \in I_\alpha\}.$$

Now set $\widehat{\mathcal{I}} = \{\widehat{I_\alpha} : \alpha \in \kappa\}$. Clearly $\widehat{\mathcal{I}}$ has cardinality 2^λ, then the only thing left to prove is that it is an $\mathcal{J}$-independent family.

Fix $s; 2^\lambda \to 2$, with $|s| < \omega$. We want to see that $\widehat{\mathcal{I}}^s \in \mathcal{J}^+$. As $\mathcal{I}$ is independent, $\mathcal{I}^s \neq \emptyset$ and moreover $\beta \in \mathcal{I}^s$ implies that $X_\beta \subseteq \widehat{I_\alpha}$ for all α such that $s(\alpha) = 0$ and $X_\beta \cap \widehat{I_\alpha} \in \mathcal{J}$ for all α such that $s(\alpha) = 1$, i.e., $X_\beta \subseteq_{\mathcal{J}} \widehat{\mathcal{I}}^s$ and, since $X_\beta \in \mathcal{J}^+$, it follows that $\widehat{\mathcal{I}}^s \in \mathcal{J}^+$. $\square$

Next we will point out some relationships between the non-existence of strongly $\mathcal{J}$-independent families and the existence of large cardinals.

Definition 3.3. If $\mathcal{J}$ is an ideal on κ, we say that $\mathcal{J}$ is κ-*complete* if $\bigcup \mathcal{H} \in \mathcal{J}$, for every subfamily $\mathcal{H} \subseteq \mathcal{J}$ such that $|\mathcal{H}| < \kappa$.

[7] Such a family exists in ZFC, i.e., it is not necessary to assume any large cardinal hypotheses about λ. This can be consulted in [4] (Theorem 4.2).

Theorem 3.4. *[10] Suppose that $\mathfrak{J}$ is a κ-complete ideal on κ.*

(1) (Tarski [16]) If $\mathfrak{J}$ is λ-saturated with $2^{<\lambda} < \kappa$, then κ is measurable.
(2) (Levy-Silver [10]) If $\mathfrak{J}$ is κ-saturated and κ is weakly compact, then κ es measurable.
(3) (Kurepa [12]) If $\mathfrak{J}$ is λ-saturated with $\lambda < \kappa$, then κ has the tree property.

Corollary 3.5. *Suppose that $\mathfrak{J}$ is a κ-complete ideal on κ.*

(1) If $\lambda < \kappa$, $2^{<\lambda} < \kappa$ and it does not exists a $\mathfrak{J}$-independent family of cardinality 2^λ, then κ es measurable.
(2) If there is no $\mathfrak{J}$-independent family of cardinality 2^κ and κ es weakly compact, then κ es measurable.
(3) If $\lambda < \kappa$ and there is no $\mathfrak{J}$-independent family of cardinality 2^λ, then κ has the tree property.

Proof. We will only prove the first part, the other two parts are analogous.

Since there is no $\mathfrak{J}$-independent family of cardinality 2^λ, then, by Lemma 3.2, we have that $\mathrm{sat}(\mathfrak{J}) \leq \lambda$, i.e., $\mathfrak{J}$ is λ-saturated, then by the first part of Theorem 3.4 we have the desired result. □

Saturation of the ideal $\mathfrak{J}$ is related to the existence of strongly $\mathfrak{J}$-independent families.

Proposition 3.6. *Let $\mathfrak{J}$ be an ideal on κ and suppose that there exists a strongly $\mathfrak{J}$-independent family of cardinality κ. Then $\mathrm{sat}(\mathfrak{J}) \geq \kappa$. Furthermore, if κ is regular then κ is strongly inaccessible.*

Proof. Let $\mathcal{I}$ be a strongly $\mathfrak{J}$-independent family of cardinality κ, $\lambda < \kappa$ and $\mathcal{I}_\lambda \subseteq \mathcal{I}$ such that $|\mathcal{I}_\lambda| = \lambda$. Then for every $h : \lambda \to 2$, we have that $\mathcal{I}_\lambda^h \in \mathfrak{J}^+$ and if $h \neq g$ then $\mathcal{I}_\lambda^h \cap \mathcal{I}_\lambda{}^g = \emptyset$, which proves that $\mathrm{sat}(\mathfrak{J}) > 2^\lambda > \lambda$, and it finishes the proof. □

The method in the previous proof has the advantage that it illustrates the fact that κ is a strong limit cardinal, however the existence of a strongly $\mathfrak{J}$-independent family of cardinality κ says even more about the saturation of $\mathfrak{J}$: If $\mathfrak{J}$ is an ideal on κ and there is a strongly $\mathfrak{J}$-independent family $\mathcal{I}$ with cardinality κ, then $\mathrm{sat}(\mathfrak{J}) > \kappa$. Indeed, suppose that $\mathcal{I} = \{X_\alpha : \alpha \in \kappa\}$ and for every $\beta \in \kappa$ let $Y_\beta = X_\beta \setminus \bigcup_{\alpha\in\beta} X_\alpha$. Note that, since $\mathcal{I}$ is strongly $\mathfrak{J}$-independent, $Y_\beta \in \mathfrak{J}^+$, and if $\beta < \gamma < \kappa$ then $Y_\beta \cap Y_\gamma = \emptyset$. This proves that $\mathfrak{J}$ is not κ-saturated (since $\{Y_\beta : \beta \in \kappa\}$ is a witness of that).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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