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Q-SETS AND NORMALITY OF Ψ-SPACES

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Abstract. We tie up some loose ends in the relationship between the normality of Ψ-spaces and the existence of Q-sets.

1. Introduction

The starting point for our considerations is the following classical theorem giving a definitive solution to the separable case of the *Normal Moore Space Problem.*

Theorem 1.1 ([9]). *The following are equivalent:*
(a) There is a Q-set.
(b) There is an uncountable normal Ψ-space.
(c) There is a separable normal non-metrizable Moore space.

For the sake of completeness, we include a sketch of the proof of the theorem later in the text. We would like to mention that the implication (a) $\Rightarrow$ (c) was originally proven by R. H. Bing in [1] when he showed that the bubble space over a Q-set is a normal space. On the other hand, R. W. Heath proved the implication (c) $\Rightarrow$ (a) in [5].

An *almost disjoint family* $\mathcal{A}$ is a family of infinite subsets of ω (or any other countable set) such that $A \cap B$ is finite for distinct

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$A, B \in \mathcal{A}$. The Ψ-*space* $\Psi(\mathcal{A})$ associated with $\mathcal{A}$ is $\omega \cup \mathcal{A}$ where the points of ω are isolated and the basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$, where $F \subseteq \omega$ is finite. The space $\Psi(\mathcal{A})$ is a first countable, separable, locally compact Moore space.

An almost disjoint family $\mathcal{A}$ is said to be $\mathbb{R}$-*embeddable* if there exists a continuous $f : \Psi(\mathcal{A}) \to \mathbb{R}$ such that f is an injective function and $f(A)$ is an irrational number, while $f(n)$ is a rational one for all $A \in \mathcal{A}$ and all $n \in \omega$. For an $\mathbb{R}$-embeddable family $\mathcal{A}$, we keep the notation $X_{\mathcal{A}} = \{x_A : A \in \mathcal{A}\}$, where $f(A) = x_A$ and f is a witness that $\mathcal{A}$ is $\mathbb{R}$-embedded.

In a conversation with members of the Toronto Set Theory seminar, the following question was raised:

Question 1. If $\mathcal{A}$ is an uncountable $\mathbb{R}$-embeddable family, is $\Psi(\mathcal{A})$ normal if and only if $X_{\mathcal{A}}$ is a Q-set?

There seemed to be some confusion between the question and an analogous question for subfamilies of the Cantor tree (see Proposition 2.2). We will show that, consistently, neither of the implications is true. We first show that in a model of W. G. Fleissner and A. W. Miller, there is an $\mathbb{R}$-embeddable family $\mathcal{A}$ with $\Psi(\mathcal{A})$ normal, but $X_{\mathcal{A}}$ is not a Q-set [4]. Then we modify their method to show that there is a model of ZFC where there is an $\mathbb{R}$-embeddable family $\mathcal{A}$ for which $X_{\mathcal{A}}$ is a Q-set, yet $\Psi(\mathcal{A})$ is not normal.

2. Preliminaries

Our terminology is mostly standard: $A \subseteq^* B$ means that A is almost contained in B; that is, $A \setminus B$ is finite, $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$. For functions $f, g \in \omega^\omega$, we write $f \leq^* g$ to mean that there is some $m \in \omega$ such that $f(n) \leq g(n)$ for all $n \geq m$. The bounding number in ω^ω, $\mathfrak{b}$, is the least cardinal of an $\leq^*$-unbounded family of functions. The dominating number in ω^ω, $\mathfrak{d}$, is the least cardinal of a $\leq^*$-cofinal family of functions. A Q-*set* is an uncountable set X of reals such that every subset of X is F_σ in X. A λ-set is an uncountable set X of reals such that every countable subset of X is G_δ in X. ZFC suffices to construct a λ-set of size $\mathfrak{b}$. We say that a subset A of $\mathbb{R}$ is *concentrated* on a set $C \subseteq \mathbb{R}$ if $A \setminus U$ is countable for every open set U containing C. The set 2^ω is equipped with the product topology, that is the topology with basic open sets of the form $[s] = \{x \in 2^\omega : s \subseteq x\}$, where $s \in 2^{<\omega}$. The topology of $\mathcal{P}(\omega)$ is that obtained via the identification of each subset of ω with its characteristic function. Given a set $E \subseteq 2^\omega$, let $\widehat{E} = \{x \upharpoonright n : x \in E,\ n \in \omega\} \subseteq 2^{<\omega}$. See [3] for undefined topological notions and [6] for set theoretical notions.

The following proposition is standard and easy to prove.

Proposition 2.1. $\Psi(\mathcal{A})$ is a normal space if and only if for every $\mathcal{B} \subseteq \mathcal{A}$ there is a $J \subseteq \omega$ such that

$$\mathcal{B} = \{A \in \mathcal{A} : A \subseteq^* J\} \quad \text{and } \mathcal{A} \setminus \mathcal{B} = \{A \in \mathcal{A} : A \cap J =^* \emptyset\}.$$

The set J in the conclusion of this proposition is called *partitioner* for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$. Notice that if $\mathcal{B} = \{A \in \mathcal{A} : A \subseteq^* J\}$, then

$$\mathcal{B} = \bigcup_{n\in\omega} \bigcap_{m\in\omega} \{A \in \mathcal{A} : m \in A \setminus n \Rightarrow m \in J\};$$

that is, $\mathcal{B}$ is an F_σ-set of $\mathcal{A}$ as subspace of $\mathcal{P}(\omega)$. Therefore, if $\Psi(\mathcal{A})$ is a normal space then $\mathcal{A}$ is a Q-set (as subspace of $\mathcal{P}(\omega)$).

For $x \in 2^\omega$, put $A_x = \{x \upharpoonright n : n \in \omega\}$ and for a subset X of 2^ω let $\mathcal{A}_X = \{A_x : x \in X\}$. Then $\mathcal{A}_X$ is an almost disjoint family of subsets of $2^{<\omega}$. Call this $\mathcal{A}_X$ the *almost disjoint family corresponding to X*.

Proposition 2.2 (Folklore)**.** Let $X \subseteq 2^\omega$ and $\mathcal{A}_X$ be the almost disjoint family corresponding to X. Then X is a Q-set if and only if $\Psi(\mathcal{A}_X)$ is a normal space.

Proof: Assume that X is a Q-set and $\mathcal{B} \subseteq \mathcal{A}$, and let $B = \{x \in X : A_x \in \mathcal{B}\}$. Since X is a Q-set there are closed subsets F_n and G_n of X such that $B = \bigcup_{n\in\omega} F_n$ and $X \setminus B = \bigcup_{n\in\omega} G_n$. Define $J_o = \widehat{F}_0$, $K_0 = \widehat{G}_0 \setminus \widehat{F}_0$, and $J_n = \widehat{F}_n \setminus \bigcup_{i<n} \widehat{G}_i$ as well as $K_n = \widehat{G}_n \setminus \bigcup_{i\leq n} \widehat{F}_n$ for $n > 0$. Put $J = \bigcup_{n\in\omega} J_n$ and observe that $J \cap K_m =^* \emptyset$ for every $m \in \omega$. If $A_x \in \mathcal{B}$, then there is some $n \in \omega$ such that $x \in F_n$. Moreover, since each G_i is closed in X and $G_i \cap B = \emptyset$, for $i < n$, there is some $k \in \omega$ such that $[x \upharpoonright k] \cap \bigcup_{i<n} G_i = \emptyset$. This implies that $A_x \subseteq^* J_n \subseteq J$. Similarly, if $x \in X \setminus B$ there are $k, m \in \omega$ such that $x \in G_m$ and $[x \upharpoonright k] \cap \bigcup_{i\leq n} F_i = \emptyset$; this implies $A_x \cap J =^* \emptyset$. By Proposition

2.1 this suffices to show that $\Psi(\mathcal{A})$ is normal. The other direction follows from the comments following Proposition 2.1. $\square$

Proof of Theorem 1.1: The equivalence between (a) and (b) follows from Proposition 2.2. For (b) implies (c) we need only to recall that $\Psi(\mathcal{A})$ is always a separable Moore space which is non-metrizable if $\mathcal{A}$ is uncountable. To show that (c) implies (b), first observe that a separable normal Moore space which is not metrizable has an uncountable closed discrete subset. This is a consequence of Bing's theorem that a Moore space X is metrizable if and only if X is collectionwise normal. So, let X be a separable non-metrizable normal Moore space, let D be a countable dense subset of X, and let F be a closed discrete uncountable subset of X disjoint from D. Taking sequences of elements in D which converge to the points in F, we obtain an uncountable almost disjoint family $\mathcal{A}$. The normality of X implies that of $\Psi(\mathcal{A})$. $\square$

Restricting ourselves to $\mathbb{R}$-embeddable almost disjoint families does not impose any restriction due to the following simple observation.

Proposition 2.3. Every almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is normal is $\mathbb{R}$-embeddable.

Proof: If $\Psi(\mathcal{A})$ is a normal space then we can take any injection $g : \mathcal{A} \to \mathbb{R} \setminus \mathbb{Q}$ and use the Tietze extension theorem to find a continuous extension $\overline{g} : \Psi(\mathcal{A}) \to \mathbb{R}$ of g. Now $\overline{g}$ can be modified to obtain a function f witnessing that $\mathcal{A}$ is $\mathbb{R}$-embedded; simply let $f \restriction \mathcal{A} = \overline{g} \restriction \mathcal{A}$ and, for $n \in \omega$, $f(n) \in \mathbb{Q} \setminus \{f(0), \dots, f(n-1)\}$ such that $|f(n) - \overline{g}(n)| < \frac{1}{n}$. The continuity of f follows from that of $\overline{g}$. $\square$

3. Main results

In this section we describe how to construct models of ZFC where either of the possible implications in Question 1 fails. With a little bit of care one can actually construct a single model where both implications fail.

Even though the belief in a positive answer for Question 1 was not correct, it was partially supported by the next theorem.

Theorem 3.1 (Folklore). $\mathsf{MA}_{\sigma\text{-centered}}$ *implies that for an* $\mathbb{R}$*-embeddable almost disjoint family* $\mathcal{A}$, $\Psi(\mathcal{A})$ *is normal if and only if* $X_{\mathcal{A}}$ *is a* Q*-set.*

Proof: For the forward implication just recall that by Jones' Lemma, the cardinality of $\mathcal{A}$ must be less than $\mathfrak{c}$ and by a result of J. Silver (under $\mathsf{MA}_{\sigma\text{-centered}}$) every set of reals of size less than $\mathfrak{c}$ is a Q-set. The reverse implication follows directly from the next lemma. □

Lemma 3.2. *Suppose that* $\mathcal{A}$ *is an* $\mathbb{R}$*-embeddable family such that* $X_{\mathcal{A}}$ *is a* Q*-set. If* $|\mathcal{A}| < \mathfrak{b}$*, then* $\Psi(\mathcal{A})$ *is a normal space.*

Proof: For convenience we shall use the *bubble space* to show that $\Psi(\mathcal{A})$ is normal. It is an old result of Bing's that the bubble space over a Q-set is a normal space. See [10, p. 709].

Let $\varphi : \Psi(\mathcal{A}) \to \mathbb{R}$ witness that $\mathcal{A}$ is $\mathbb{R}$-embedded, let $X_{\mathcal{A}} = \{x_A : A \in A\}$, and put $q_n = \varphi(n)$, for every $n \in \omega$. By hypothesis $X_{\mathcal{A}}$ is a Q-set. In order to establish that $\Psi(\mathcal{A})$ is normal, let $\mathcal{B} \subseteq \mathcal{A}$. As the bubble space over $X_{\mathcal{A}}$ is normal, there are basic neighborhoods B_x, for $x \in X_{\mathcal{A}}$, such that $U_{\mathcal{B}} = \bigcup\{B_{x_B} : B \in \mathcal{B}\}$ and $W_{\mathcal{B}} = \bigcup\{B_{x_A} : A \in \mathcal{A} \setminus \mathcal{B}\}$ are disjoint.

It is clear that, for $A \in \mathcal{A}$, there is a function $f_A : A \to \omega$ satisfying

(1) f_A is non-decreasing,
(2) f_A is finite-to-one, and
(3) eventually $\left\langle q_n, \frac{1}{f_A(n)} \right\rangle \in B_{x_A}$.

The idea is that the function f_A can be used to lift the points in A to the neighborhood B_{x_A}. Define $g_A : \omega \to \omega$ by

$$g_A(n) = \max\{a \in A : f_A(a) \leq n\}.$$

Since $|\mathcal{A}| < \mathfrak{b}$, there is an increasing function $g : \omega \to \omega$ such that $g_A \leq^* g$ for all $A \in \mathcal{A}$. Let $f : \omega \to \omega$ be defined by

$$f(n) = \max\{k \in \omega : g(k) \leq n\}$$

for every $n \in \omega$. Then $f \restriction A \leq^* f_A$ for every $A \in \mathcal{A}$ and f is finite to one. This f can be used to do the lifting of the points q_n in a uniform way for every $A \in \mathcal{A}$; i.e., for large $n \in A$, $\left\langle q_n, \frac{1}{f(n)} \right\rangle \in B_{x_A}$ for every $A \in \mathcal{A}$.

Then the partitioner for $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ will be

$$J = \left\{ n \in \omega : \left\langle q_n, \frac{1}{f(n)} \right\rangle \in U_B \right\}.$$

It is clear that $B \in \mathcal{B} \Rightarrow B \subseteq^* J$ and $A \in \mathcal{A} \setminus \mathcal{B} \Rightarrow A \cap J =^* \emptyset$. $\square$

Now we show that there is a model in which the implication "if $\Psi(\mathcal{A})$ is an uncountable normal space then $X_{\mathcal{A}}$ is Q-set in $\mathbb{R}$" fails.

Theorem 3.3. It is consistent that there is an uncountable $\mathcal{A}$ which is $\mathbb{R}$-embeddable and $\Psi(\mathcal{A})$ is normal but $X_{\mathcal{A}}$ is not a Q-set.

Proof: In [4], a model is described where there is a Q-set X which consists of the irrationals concentrated on some countable set F of the irrationals disjoint from X. Partition $\mathbb{Q}$ into two disjoint dense sets D_0 and D_1. By propositions 2.2 and 2.3, there is an $\mathbb{R}$-embedded almost disjoint family $\mathcal{A}_0$ of subsets of D_0 such that $X_{\mathcal{A}_0} = X$. Consider also the almost disjoint family $\mathcal{A}_1$ obtained by taking convergent sequences of elements in D_1 to the elements of F. Since F is countable, $\Psi(\mathcal{A}_1)$ is normal. Letting $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$, $\Psi(\mathcal{A})$ is normal as it is the topological sum of $\Psi(\mathcal{A}_0)$ and $\Psi(\mathcal{A}_1)$, both of which are normal. However, as X is concentrated on F, $X_{\mathcal{A}} = X \cup F$ cannot be a Q-set.[1] $\square$

Theorem 3.4. *It is consistent that there is an almost disjoint family $\mathcal{A}$ that is $\mathbb{R}$-embeddable, $X_{\mathcal{A}}$ is a Q-set, yet $\Psi(\mathcal{A})$ is not a normal space.*

Proof: We will use a finite support iteration $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \right\rangle$ of c.c.c. forcings that will be described next. The idea is that $\mathbb{P}_0$ generically adds an $\mathbb{R}$-embeddable family $\mathcal{A}$ and the rest of the iteration makes $X_{\mathcal{A}}$ a Q-set without making $\Psi(\mathcal{A})$ a normal space.

For technical reasons, fix families $\mathcal{W}_n$, $n \in \omega$, of open intervals with irrational endpoints such that

- the length of each interval in $\mathcal{W}_n$ is at most $\frac{1}{n}$, and
- $\mathcal{W}_{n+1}$ refines $\mathcal{W}_n$.

[1] From the referee's report: "The example of a normal $\mathbb{R}$-embedded $\Psi(\mathcal{A})$ such that $X_{\mathcal{A}}$ is not a Q-set is a little unsatisfying, in that it is homeomorphic to the topological sum of two $\Psi(\mathcal{A})$'s, each over a Q-set (one countable). I wonder if every example must be like that. Surely there would at least have to be an uncountable subset of $X_{\mathcal{A}}$ which is a Q-set."

Then $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$ is a base for the rationals with the property that, given two elements of $\mathcal{W}$, either one is contained in the other or they are disjoint. In particular, for every $q \in \mathbb{Q}$ and $n \in \omega$ there is a unique $W = W(q, n) \in \mathcal{W}_n$ such that $q \in W$.

Define $\mathbb{P}_0$ by $p \in \mathbb{P}_0$ if and only if $p = (a, k) \in \mathrm{Fn}(\omega_1 \times \omega, \mathbb{Q}) \times \omega$. For $p = (a_p, k_p)$ and $q = (a_q, k_q)$ declare $p \le q$ if and only if

(1) $a_p \supseteq a_q$ and $k_p \ge k_q$, and
(2) if $a_q(\alpha, \_)$ and $a_p(\alpha, \_)$ are the obvious functions and n_q^α is the largest element in the domain of $a_q(\alpha, \_)$ then

$$a_p(\alpha, n) \in W\left(a_q(\alpha, n_q^\alpha), k_q\right)$$

for all $n \in \mathrm{dom}(a_p(\alpha, \_)) \setminus \mathrm{dom}(a_q(\alpha, \_))$. Moreover, this holds for all $\alpha \in \omega_1$ such that there is some $n \in \omega$ for which $(\alpha, n) \in \mathrm{dom}(a_q)$.[2]

If G_0 is a generic filter in $\mathbb{P}_0$, then G_0 codes a family of sequences $a_\alpha : \omega \to \mathbb{Q}$ given by $a_\alpha(n) = a_p(\alpha, n)$ for some $p \in G_0$. Thus, $\mathbb{P}_0$ actually adds ω_1 Cohen reals—the limits of the Cauchy sequences $\langle a_\alpha(n) : n \in \omega\rangle$. It is clear that letting $A_\alpha = \{a_\alpha(n) : n \in \omega\}$, we have an $\mathbb{R}$-embedded almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ in $V[G_0]$.

Let $\mathcal{B} = \{B_n : n \in \omega\}$ be a base for the topology of $\mathbb{R}$. For $Y \subseteq X_{\mathcal{A}}$, let $\mathbb{P}(Y)$ be the set of all r such that

(1) r is finite subset of $\omega \times (\mathcal{B} \cup Y)$;
(2) for $B \in \mathcal{B}$ and $x \in Y$, the set r satisfies $\langle n, B\rangle \in r$, and $\langle n, x\rangle \in r$ implies $x \notin B$.

The ordering on $\mathbb{P}(Y)$ is just $r \le r'$ if and only if $r \supseteq r'$. This forcing is defined in [4].

The forcing $\mathbb{P}(Y)$ makes Y an F_σ subset of $X_{\mathcal{A}}$. Indeed, if G is a generic filter in $\mathbb{P}(Y)$ over a model V, then set for each $n \in \omega$,

$$U_n = \{x \in X_{\mathcal{A}} : (\exists r \in G)\,(\exists B \in \mathcal{B})\,(\langle n, B\rangle \in r \ \wedge\ x \in B)\},$$

and set $K = \bigcup\{X \setminus U_n : n \in \omega\}$. Then each U_n is open relative to $X_{\mathcal{A}}$ and K is an F_σ relative to $X_{\mathcal{A}}$. To see that $K = Y$, proceed as follows. For all $x \in Y$ and $r \in \mathbb{P}(Y)$, there are $r' \supseteq r$ and $n \in \omega$ such that $\langle n, x\rangle \in r'$. Then $x \in X \setminus U_n \subseteq K$. On the other hand, if $x \in X_{\mathcal{A}} \setminus Y$, $r \in \mathbb{P}(Y)$ and $n \in \omega$, there are $r' \supseteq r$ and $B \in \mathcal{B}$ such that $\langle n, B\rangle \in r'$ and $x \in B$. Then $x \in U_n$ for all $n \in \omega$.

[2]The point of choosing the families $\mathcal{W}_n$ is to make the order of $\mathbb{P}_0$ transitive.

Assume the ground model V is a model of CH. Using the usual bookkeeping techniques, we can take $\mathbb{Q}_\alpha$ to be such that $\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha = \mathbb{P}(\dot{Y}_\alpha)$ where $\dot{Y}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $X_{\mathcal{A}}$ and doing this in such a way that every subset of $X_{\mathcal{A}}$ eventually appears as some Y_α. Thus, $\mathbb{P}_{\omega_2}$ will force $\mathcal{A}$ to be an $\mathbb{R}$-embedded almost disjoint family and $X_{\mathcal{A}}$ to be a Q-set. Fix a generic $G \subseteq \mathbb{P}_{\omega_2}$ over the ground model V.

To show that $\Psi(\mathcal{A})$ is not normal in $V[G]$, it suffices to find a $\mathcal{C} \subseteq \mathcal{A}$ such that $\mathcal{C}$ and $\mathcal{A} \setminus \mathcal{C}$ cannot be separated. Consider simply $\mathcal{C} = \{A_\alpha : \alpha \in \omega_1 \text{ is a limit ordinal}\}$. Suppose $J \subseteq \mathbb{Q}$ is such that

$$\mathcal{C} = \{A \in \mathcal{A} : A \subseteq^* J\} \quad \text{and} \quad \mathcal{A} \setminus \mathcal{C} = \{A \in \mathcal{A} : A \cap J =^* \emptyset\}.$$

By genericity, J is a dense and co-dense subset of the rationals. The conclusion will then follow from the next lemma. $\square$

Lemma 3.5. *In $V[G]$, the set $\{\alpha \in \omega_1 : A_\alpha \subseteq^* J\}$ is at most countable for every dense and co-dense $J \subseteq \mathbb{Q}$.*

To prove the lemma we first show that $\mathbb{P}_{\omega_2}$ is "semi-Cohen" (see [2]).

Lemma 3.6. *If $p \in \mathbb{P}_{\omega_2}$ and M is a countable elementary submodel of some big enough $H(\theta)$ such that $\mathbb{P}_{\omega_2} \in M$, then there is a $\overline{p} \in \mathbb{P}_{\omega_2} \cap M$ such that whenever $r \in \mathbb{P}_{\omega_2} \cap M$ is such that $r \leq \overline{p}$, r is compatible with p.*

Proof: For every $p \in \mathbb{P}_{\omega_2}$ there is some $\beta < \omega_2$ such that $p \in \mathbb{P}_\beta$; thus, the proof can be done by induction on $\beta \leq \omega_2$. For $\beta = 0$ and for β limit, the proof is easy so we only show the details for successor steps.

Suppose the lemma holds for $\mathbb{P}_\beta$ and $p \in \mathbb{P}_{\beta+1}$. There is a $p' \in \mathbb{P}_\beta$ such that

$$p' \Vdash p(\beta) = \{\langle n_i, B_i\rangle : i < n_0\} \cup \{\langle n_i, \dot{x}_{\gamma_i}\rangle : i < n_2\},$$

for some $n_0, n_1, n_2 \in \omega$ with $n_1 \leq n_2$, and such that $\{\gamma_i : i < n_1\} \subseteq M$ while $\{\gamma_i : n_1 \leq i < n_2\} \cap M = \emptyset$.

By induction hypothesis there exists a $\overline{p'} \in \mathbb{P}_\beta \cap M$ satisfying the conclusion for p' instead of p. Find $\{\gamma_i' : n_1 \leq i < n_2\} \subseteq M$ disjoint from $\{\gamma_i : i < n_2\}$ and define $\overline{p} \in M$ as follows:

$\overline{p}(0) = \overline{p'}(0) \cup \{\langle\langle\gamma_i', m, q_j\rangle, n\rangle : \langle\langle\gamma_i, m, q_j\rangle, n\rangle \in p'(0) \wedge i < n_2\}$;

$\overline{p}(\alpha) = \overline{p'}(\alpha)$ for $0 < \alpha < \beta$; and

$\overline{p}(\beta) = (p(\beta) \cap M) \cup \left\{ \left\langle n_i, \dot{x}_{\gamma_i'} \right\rangle : i \in n_2 \right\}.$

Then $\overline{p}$ satisfies the requirements. Indeed, if $r \in \mathbb{P}_{\beta+1} \cap M$ and $r \leq \overline{p}$, then $r \upharpoonright \beta$ is compatible with $p \upharpoonright \beta$; thus, p and r can only be incompatible if r forces $\dot{x}_{\gamma_i}$ inside some B used in $p(\beta)$, for some $i \in n_2$. However, r (and p) can only decide a finite initial part of $\dot{A}_{\gamma_i}$ of which $\dot{x}_{\gamma_i}$ is the limit. Then there must exist $q_m \in \mathbb{Q}$ and $k \in \omega$ such that the interval $W(q_m, k)$ is contained in B, $r \Vdash q_m \in \dot{A}_{\gamma_i}$, and any mutual extension of p and r forces the new points of $\dot{A}_{\gamma_i}$ to be in $W(q_m, k)$. Since the same initial part of $\dot{A}_{\gamma_i}$ that p uses is also used by $\overline{p}$ to determine another point $\dot{x}_{\gamma_i'}$, r would also force q_m inside $\dot{A}_{\gamma_i'}$ and promise that the new points for $\dot{A}_{\gamma_i'}$ are in $W(q_m, k)$. In other words, r would force $\dot{x}_{\gamma_i'}$ inside B as well. This is a contradiction as $\overline{p} \Vdash \dot{x}_{\gamma_i'} \notin B$. □

Now we are ready to give the proof of the main lemma.

Proof of Lemma 3.5: Fix $J \subseteq \mathbb{Q}$ as in the statement of the lemma and let $\dot{J}$ be a $\mathbb{P}_{\omega_2}$-name for J. Let $M \prec H(\theta)$ be countable, with θ being a large enough regular cardinal, $\dot{J} \in M$, and M containing everything that is relevant for the proof of the lemma. We now show that $\mathbb{P}_{\omega_2} \Vdash \dot{A}_\delta \not\subseteq^* \dot{J}$ for $\delta \notin M$.

Suppose not. Then there is some $\delta \notin M$ and there is $p \in \mathbb{P}_{\omega_2}$ such that, for some $n \in \omega$, $p \Vdash \dot{A}_\delta \setminus \dot{J} \subseteq \{q_0, \ldots, q_n\}$. Let $\overline{p} \in M \cap \mathbb{P}_{\omega_2}$ be as in the conclusion of Lemma 3.6. Since $\overline{p} \Vdash$ "$\mathbb{Q} \setminus \dot{J}$ is dense in $\mathbb{R}$", $\overline{p}$ can be extended inside M to some $r \in \mathbb{P}_{\omega_2} \cap M$ such that, for some $k \in \omega$ and large enough $m \in \omega$,

- $r \Vdash q_k \notin \dot{J}$;
- q_k is a fresh new rational, i.e., $q_k \notin \{q_0, \ldots, q_n\}$, and q_k is not in the range of $p(0)(\delta, \_)$; and
- $q_k \in W(q_n, m)$.

Then r is compatible with p, so we can find a common extension r' such that $r' \Vdash q_k \in \dot{A}_\delta$ which contradicts that $p \Vdash q_k \notin \dot{A}_\delta$. □

We conclude with some remarks on Problem 291 in [8] which has already been implicitly solved. First, let us recall that a space is *pseudonormal* if every pair of disjoint closed sets can be separated by disjoint open sets, provided at least one of them is countable. Peter Nyikos asks, *Is there a pseudonormal Ψ-space of cardinality $\mathfrak{d}$?* This is equivalent to asking whether there is a λ-set of size $\mathfrak{d}$.

For if $\Psi(\mathcal{A})$ is a pseudonormal space, then $\mathcal{A}$ is a λ-set as subspace of $\mathcal{P}(\omega)$, and if $X \subseteq 2^\omega$ is a λ-set, then $\Psi(\mathcal{A}_X)$ is a pseudonormal space (the proof of this is analogous to the proof of Proposition 2.2). Miller [7, Theorem 22] showed that in the Cohen model (in which $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \aleph_2$), any λ-set has size $\aleph_1$.

Problem 291 also has a second part where Nyikos asks, *More generally, what is the maximum cardinality of a pseudonormal Ψ-space?* To this we have two comments.

First, it is not true that the maximum can always be attained. The maximum exists in the absence of inaccessible cardinals; however, assuming the existence of a strongly inaccessible cardinal, it is possible to construct a model of ZFC where $\mathfrak{c}$ is a limit cardinal, and for every $\kappa < \mathfrak{c}$ there is a λ-set of size κ, yet there is no λ-set of size $\mathfrak{c}$. The proof of this will appear elsewhere.

The other comment is that there does not seem to be any reasonable combinatorial upper bound. This is largely due to the fact that λ-sets are preserved by c.c.c. forcing extensions. For example, it is consistent that there is a λ-set of size $\mathfrak{c} = \aleph_2$, yet all cardinal invariants in the Cichoń diagram are equal to $\aleph_1$.

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