Pseudocompactness of hyperspaces

Michael Hrušák [1]

Instituto de Matemáticas, UNAM, Morelia, Mich. México.

Fernando Hernández-Hernández [2]

Facultad de Ciencias Físico Matemáticas, UMSNH Morelia, Mich. México.

Iván Martínez-Ruiz [3]

Instituto de Física y Matemáticas, UMSNH Morelia, Mich. México.

Abstract

We consider the following question of Ginsburg: *Is there any relationship between the pseudocompactness of X^ω and that of the hyperspace 2^X?* We do that first in the context of Mrówka-Isbell spaces $\Psi(\mathcal{A})$ associated with a maximal almost disjoint (MAD) family $\mathcal{A}$ on ω answering a question of J. Cao and T. Nogura. The space $\Psi(\mathcal{A})^\omega$ is pseudocompact for every MAD family $\mathcal{A}$. We show that

(1) $(\mathfrak{p} = \mathfrak{c})$ $2^{\Psi(\mathcal{A})}$ is pseudocompact for every MAD family $\mathcal{A}$.

(2) $(\mathfrak{h} < \mathfrak{c})$ There is a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is not pseudocompact.

We also construct a ZFC example of a space X such that X^ω is pseudocompact, yet 2^X is not.

AMS Classification: Primary 54C65, 54B20, 03E0.

Key words: Hyperspace, Vietoris topology, Ψ-space, Mrówka-Isbell space, MAD family, pseudocompact.

---

[1] E-mail: `michael@matmor.unam.mx`
[2] E-mail: `fernando@matmor.unam.mx`
[3] E-mail: `ivan@matmor.unam.mx`
[4] Authors gratefully acknowledge support from PAPIIT grant IN106705 and CONACyT grant 46337. The first author's research was also partially supported by GA ČR grant 201-03-0933. Third author is supported by the Consejo Nacional de Ciencia y Tecnología CONACyT, México. Scholarship 189295.

1 Introduction

The hyperspace of a space X (denoted by 2^X) consists of all closed nonempty subsets of X. We consider 2^X equipped with the *Vietoris topology*, i.e. the topology generated by sets of the form:

$$\langle U; V_0, \ldots, V_n\rangle = \left\{F \in 2^X : F \subseteq U \text{ and } F \cap V_i \neq \emptyset \text{ for every } i \leq n\right\},$$

where $U, V_0, \ldots, V_n$ are nonempty open subsets of X. The equivalence between the compactness of a space and its hyperspace was established by Michael and Vietoris. It is natural to ask if there is a similar relationship with respect to weaker compactness type properties, in particular, with respect to countable compactness and pseudocompactness. Recall that a space X is *pseudocompact* if every continuous real-valued function f on X is bounded.[5] J. Ginsburg obtained some partial results:

Theorem 1.1 ([Gin75]) *(a) If every power of a space X is countably compact then so is 2^X.*

(b) If 2^X is countably compact (pseudocompact) then so is every finite power of X.

Ginsburg also presented an example of a completely regular space such that every finite power is countably compact, hence pseudocompact, but whose hyperspace is not pseudocompact. He asked: *Is there any relationship between the pseudocompactness of X^ω and that of the hyperspace 2^X*? His question was brought to the our attention by J. Cao, T. Nogura and A. Tomita and their article [CNT04] in which they considered Ginsburg's question and gave the following partial answer:

Theorem 1.2 ([CNT04]) *If X is a homogeneous Tychonoff space such that 2^X is pseudocompact then X^ω is pseudocompact.*

J. Cao and T. Nogura, in a private conversation, explicitly asked whether 2^X is pseudocompact for some/every Mrówka-Isbell space X.

In this note we answer Cao and Nogura's question by showing that $\mathfrak{p} = \mathfrak{c}$ implies that $2^{\Psi(\mathcal{A})}$ is a pseudocompact space for every MAD family $\mathcal{A}$, while

[5] Many authors (e.g. [Eng89]) consider pseudocompactness only for Tychonoff spaces. However, most of our hyperspaces are not Tychonoff. In fact, a hyperspace 2^X is a Tychonoff space if and only if X is normal (see [Eng89, p. 121]). For Tychonoff spaces, pseudocompactness is equivalent to the fact that there is no infinite discrete family of open subsets of X. This is in general not true for Hausdorff spaces. Nevertheless, in the context of hyperspaces all the "standard" definitions of pseudocompactness coincide (see [Gin75]).

$\mathfrak{h} < \mathfrak{c}$ implies that there is a MAD family $\mathcal{A}$ for which $2^{\Psi(\mathcal{A})}$ is not pseudocompact. We also construct a ZFC example of a space X such that X^ω is pseudocompact, yet 2^X is not pseudocompact. This answers Ginsburg question in one direction and shows that the equivalent of (a) of Theorem 1.1 does not hold for pseudocompactness.

The set-theoretic notation we use is standard and follows e.g. [Kun83], [Jec97] and [BJ95]. The symbols $\mathfrak{p}$, $\mathfrak{h}$ and $\mathfrak{c}$ refer to well-known cardinal invariants of the continuum; $\mathfrak{c}$ denotes the cardinality of the continuum, $\mathfrak{p}$ and $\mathfrak{h}$ will be defined below. We refer to [vD84] and [BJ95] for more information.

2 Preliminaries

An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint (AD)* if every two distinct elements of $\mathcal{A}$ have finite intersection. A family $\mathcal{A}$ is *maximal almost disjoint* (MAD) if it is almost disjoint and maximal with this property.

Definition 2.1 Let $\mathcal{A}$ be an AD family. The *Mrówka-Isbell space* $\Psi(\mathcal{A})$ associated to $\mathcal{A}$ is defined as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated, and basic neighbourhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a first countable, locally compact space. If $\mathcal{A}$ is infinite then $\Psi(\mathcal{A})$ is not countably compact and $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is a MAD family, see e.g. [vD84].

Recall that a subset A of topological space X is *relatively countably compact* in X if every $E \in [A]^\omega$ has an accumulation point in X. One way to show that a space is pseudocompact is to show that it has a dense relatively countably compact subspace. The following lemma is easy to prove.

Lemma 2.2 *Let X have a dense set D of isolated points. Then the following are equivalent:*

(1) X is pseudocompact.
(2) D is relatively countably compact in X.

A subset A of a topological space X is *relatively sequentially compact* in X if every sequence of elements of A has a subsequence which is convergent in X. If X has a dense set which is even relativelly sequentially compact then all powers of X are pseudocompact.

Proposition 2.3 *If X has a dense subset which is relatively sequentially compact in X, then X^ω is pseudocompact.*

Proof. Assume $D \subseteq X$ is relatively sequentially compact and dense. The same proof (see e.g. [vD84, Theorem 6.9]) that shows that countable product of sequentially compact spaces is sequentially compact, shows that D^{ω} is a dense relatively sequentially compact subspace of X^{ω}, which directly implies pseudocompactness of X^{ω}. ■

As a direct consequence we get:

Lemma 2.4 $(\Psi(\mathcal{A}))^{\omega}$ *is pseudocompact for every MAD family* $\mathcal{A}$.

Let Fin denote the set of all non-empty finite subsets of ω. The following lemma, which is easy to prove, will be used frequently.

Lemma 2.5 *If* X *is a topological space such that* ω *is the dense set of isolated points of* X*, then* Fin *is a dense set of isolated points in* 2^X.

3 Hyperspaces of Mrówka-Isbell spaces under $\mathsf{MA}_{\sigma\text{-centered}}$

In this section we show that it is consistent with ZFC that $2^{\Psi(\mathcal{A})}$ is pseudocompact for every MAD family $\mathcal{A}$.

We say that a set A is *almost contained* in B, $A \subseteq^* B$, if $A \setminus B$ is finite. $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$. Recall that a family $\mathcal{F} \subseteq [\omega]^{\omega}$ is *centered* if the intersection of any finite subset of $\mathcal{F}$ is infinite. The *pseudo-intersection number* $\mathfrak{p}$ is defined as the minimal size of a centered family $\mathcal{F} \subseteq [\omega]^{\omega}$ without a pseudo-intersection, i.e. the minimal size of a centered family $\mathcal{F} \subseteq [\omega]^{\omega}$ such that for every $A \in [\omega]^{\omega}$ there is an $F \in \mathcal{F}$ such that $A \setminus F$ is infinite. By a theorem of M. G. Bell [Bel81], the assumption $\mathfrak{p} = \mathfrak{c}$ is equivalent to Martin´s Axiom for σ-centered partial orders.

We introduce the following notation: Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an almost disjoint family. Given a one-to-one sequence $Y = \langle F_n : n \in \omega \rangle \subseteq \mathrm{Fin}$ and $A \subseteq \omega$, let

- $I_A = \{n \in \omega : A \cap F_n \neq \emptyset\}$,
- $M_A = \{n \in \omega : F_n \subseteq A\}$.

If, moreover, $F \in 2^{\Psi(\mathcal{A})}$ let

- $\mathcal{F}_F = \left\{I_{A \setminus k} : A \in F \cap \mathcal{A},\ k \in \omega\right\} \cup \{I_{\{n\}} : n \in F \cap \omega\}$.

The next lemma is the main technical tool for this section.

Lemma 3.1 *Given a one-to-one sequence* $Y = \langle F_n : n \in \omega \rangle \subseteq \mathrm{Fin}$ *and* $F \subseteq \Psi(\mathcal{A})$*, the following are equivalent:*

(1) *F is an accumulation point of Y in the hyperspace $2^{\Psi(\mathcal{A})}$.*
(2) *For every $P \subseteq \omega$ that satisfies $F \cap \omega \subseteq P$ and $(\forall A \in F \cap \mathcal{A})\,(A \subseteq^* P)$, the family $\mathcal{F}_F \cup \{M_P\}$ is centered.*

Note that, in particular, (2) implies that $\mathcal{F}_F$ is centered, as $P = \omega$ satisfies the hypothesis for any Y and F.

Proof. Suppose that F is an accumulation point of Y. Let $P \subseteq \omega$ be such that $F \cap \omega \subseteq P$ and $A \subseteq^* P$ for all $A \in F \cap \mathcal{A}$. Then $V = P \cup (F \cap \mathcal{A})$ is an open subset of $\Psi(\mathcal{A})$ which contains F. To see that $\mathcal{F}_F \cup \{M_P\}$ is centered let

$$\mathcal{Q} = \left\{I_{A_0 \setminus k_0}, \ldots, I_{A_m \setminus k_m}\right\} \cup \left\{I_{\{a_0\}}, \ldots, I_{\{a_l\}}\right\} \cup \{M_P\} \subseteq \mathcal{F}_F \cup \{M_P\},$$

where $A_i \in F \cap \mathcal{A},\ k_i \in \omega$ for all $i \leq m$ and $a_i \in F \cap \omega$ for all $i \leq l$. Then

$$U = \langle V; \{A_0\} \cup A_0 \setminus k_0, \ldots, \{A_m\} \cup A_m \setminus k_m, \{a_0\}, \ldots, \{a_l\}\rangle$$

is a neighbourhood of F in $2^{\Psi(\mathcal{A})}$ and therefore $Y \cap U$ is infinite. There is an $I \in [\omega]^{\omega}$ such that $(A_j \setminus k_j) \cap F_i \neq \emptyset$ and $\{a_k\} \cap F_i \neq \emptyset$ for all $i \in I$ and all $j \leq m,\ k \leq l$. By the definition of I_F and M_P, it follows that $I \subseteq \bigcap \mathcal{Q}$ and so $\mathcal{F}_F \cup \{M_P\}$ is centered.

Conversely, assume that $\mathcal{F}_F \cup \{M_P\}$ is centered and let

$$U = \langle V; \{A_0\} \cup A_0 \setminus k_0, \ldots, \{A_m\} \cup A_m \setminus k_m, \{a_0\}, \ldots, \{a_l\}\rangle$$

be a basic neighbourhood of F in $2^{\Psi(\mathcal{A})}$. Let $P = V \cap \omega$. Since $F \subseteq V$, $F \cap \omega \subseteq P$ and $A \subseteq^* P$ for all $A \in F \cap \mathcal{A}$. As $\mathcal{F}_F \cup \{M_P\}$ is centered,

$$\bigcap \left\{I_{A_i \setminus k_i} : i \leq m\right\} \cap \bigcap \left\{I_{\{a_i\}} : i \leq l\right\} \cap \{M_P\}$$

is infinite. Therefore so is $U \cap Y$. This shows that F is an accumulation point of Y. ■

Theorem 3.2 ($\mathfrak{p} = \mathfrak{c}$) *$2^{\Psi(\mathcal{A})}$ is pseudocompact for every MAD family $\mathcal{A}$.*

Proof. By Lemma 2.5 and Lemma 2.2, it suffices to show that given a MAD family $\mathcal{A}$, every one-to-one sequence $Y = \langle F_n : n \in \omega\rangle \subseteq \mathrm{Fin}$ has an accumulation point in $2^{\Psi(\mathcal{A})}$. Consider such a Y and let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^{\omega}$, where each element is listed $\mathfrak{c}$-many times and $P_0 = \omega$. Recursively construct a family $\{E_\alpha : \alpha < \mathfrak{c}\}$ such that, for every $\alpha < \mathfrak{c}$:

(1) $E_\alpha \subseteq \Psi(\mathcal{A})$,
(2) $|E_\alpha| \leq |\alpha| + \omega$,
(3) $\alpha \leq \beta$ implies $E_\alpha \subseteq E_\beta$,
(4) $\mathcal{F}_\alpha = \left\{I_{A \setminus k} : A \in E_\alpha \cap \mathcal{A},\ k \in \omega\right\} \cup \left\{I_{\{n\}} : n \in E_\alpha \cap \omega\right\}$ is centered, and
(5) one of the following occurs:

(a) $(E_\alpha \cap \omega) \setminus P_\alpha \neq \emptyset$,
(b) there is an $A \in E_\alpha \cap \mathcal{A}$ such that $A \not\subseteq^* P_\alpha$, or
(c) $\mathcal{F}_\alpha \cup \{M_{P_\alpha}\}$ is centered.

The accumulation point of Y is going to be the closure of $\bigcup_{\alpha<\mathfrak{c}} E_\alpha$. Next we will carry out the recursive construction.

There is an $A \in \mathcal{A}$, such that for every $k \in \omega$ there is an $n \in \omega$ such that $(A \setminus k) \cap F_n \neq \emptyset$. Let $E_0 = \{A\}$. As $P_0 = \omega$, (1) - (5) hold.

Assume that $0 < \alpha < \mathfrak{c}$ and that E_β has been constructed for all $\beta < \alpha$. Since $\mathcal{F}_\beta$ is centered for all $\beta < \alpha$, so is $\mathcal{F} = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. Now, if $\mathcal{F} \cup \{M_{P_\alpha}\}$ is centered, then letting $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ all properties (1) - (5) are satisfied. If $\mathcal{F} \cup \{M_{P_\alpha}\}$ is not centered, then as $\mathcal{F}$ is centered and $|\mathcal{F}| < \mathfrak{p}$, there is a $J \in [\omega]^\omega$ almost contained in all elements of $\mathcal{F}$ such that $|J \cap M_{P_\alpha}| = \emptyset$. Consider the following two cases:

Case 1: There is an $m \in \omega \setminus P_\alpha$ such that $\{n \in J : m \in F_n\}$ is infinite.

Let $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{m\}$. All clauses but (4) are evidently true. To see that the fourth clause also holds for E_α, take

$$\mathcal{G} = \{I_{A_0 \setminus k_0}, \ldots, I_{A_s \setminus k_s}, I_{\{a_0\}}, \ldots, I_{\{a_t\}}, I_{\{m\}}\} \subseteq \mathcal{F} \cup \{I_{\{m\}}\}.$$

Since $\{n \in J : m \in F_n\} = I_{\{m\}} \cap J$ is infinite and $J \subseteq^* F$ for all $F \in \mathcal{F}$, then

$$\{n \in J : m \in F_n\} \subseteq^* \bigcap_{i \leq s} I_{A_i \setminus k_i} \cap \bigcap_{i \leq t} I_{\{a_i\}} \cap I_{\{m\}},$$

thus $\bigcap \mathcal{G}$ is infinite and therefore $\mathcal{F}_\alpha$ is centered.

Case 2: $\{n \in \omega : m \in F_n\}$ is finite for all $m \in \omega \setminus P_\alpha$.

It follows that $\bigcup_{n\in\omega} F_n \setminus P_\alpha$ is infinite and hence there is $A \in \mathcal{A}$ such that $|A \cap (\bigcup_{n\in\omega} F_n) \setminus P_\alpha| = \omega$. In this case let $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{A\}$. Again, only clause (4) requires verification. Take

$$\mathcal{G} = \left\{I_{A_0 \setminus k_0}, \ldots, I_{A_s \setminus k_s}, I_{\{a_0\}}, \ldots, I_{\{a_t\}}, I_{A \setminus k}\right\} \subseteq \mathcal{F}.$$

As the set $\bigcup_{n\in\omega} F_n \setminus P_\alpha$ is infinite, so is the set $\{n \in J : A \cap F_n \neq \emptyset\}$. Moreover,

$$\{n \in J : A \cap F_n \neq \emptyset\} \subseteq J \subset^* F$$

for all $F \in \mathcal{F}$. So, $\{n \in J : A \cap F_n \neq \emptyset\} \subseteq \bigcap \mathcal{G}$ and therefore $\mathcal{F}_\alpha$ is centered. This completes the construction of the sets E_α.

Let E be the closure (in $\Psi(\mathcal{A})$) of $\bigcup_{\alpha<\mathfrak{c}} E_\alpha$. We claim that E is an accumulation point of Y in $2^{\Psi(\mathcal{A})}$. By Lemma 3.1, it suffices to show that for every $P \subseteq \omega$

one of the following holds:

(1) $(E \cap \omega) \setminus P \neq \emptyset$,
(2) there is $A \in E \cap \mathcal{A}$ such that $|A \setminus P| = \aleph_0$, or
(3) $\mathcal{F}_E \cup \{M_{P_\alpha}\}$ is centered.

First we show that $\mathcal{F}_E$ is centered. Clearly $\mathcal{F} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$ is centered and $E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha \subseteq \mathcal{A}$, since all elements of ω are isolated. Moreover, $(A \setminus k) \cap \bigcup_{\alpha<\mathfrak{c}} E_\alpha \neq \emptyset$ for each $A \in E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ and each $k \in \omega$. Thus, given any $A \in E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ and $k \in \omega$, there is an $m \in (A \setminus k) \cap \bigcup_{\alpha<\mathfrak{c}} E_\alpha$. Note that for this m we have that $I_{\{m\}} \subseteq I_{A\setminus k}$. This implies that for all $F \in \mathcal{F}_E$ there is an $G \in \mathcal{F}$ such that $G \subseteq F$. As $\mathcal{F}$ is centered, so is $\mathcal{F}_E$.

Finally, consider $P \subseteq \omega$ and assume that $\mathcal{F}_E \cup \{M_P\}$ is not centered. There are $A_0, \ldots, A_n \in E \cap \mathcal{A}$, $k_0, \ldots, k_n \in \omega$ and $m_0, \ldots, m_k \in E \cap \omega$ such that

$$\bigcap_{i \le n} I_{A_i \setminus k_i} \cap \bigcap_{i \le k} I_{\{m_i\}} \cap M_P =^* \emptyset.$$

For each $i \le n$ such that $A_i \in E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ there is an ordinal $\alpha_i < \mathfrak{c}$ and there is an $a_i \in E_{\alpha_i}$ such that $I_{\{m_i\}} \subseteq I_{A_i \setminus k_i}$, as we saw in the previous paragraph. Choose $\beta < \mathfrak{c}$ greater than all the α_i's and such that $A_j \in E_\beta$ for all elements A_j, $j \le n$, that are members of $\bigcup_{\alpha<\mathfrak{c}} E_\alpha$. Let $\alpha < \mathfrak{c}$ be such that $P = P_\alpha$ and $\alpha > \beta$. It follows that $\mathcal{F}_\alpha \cup \{M_P\}$ is not centered either. Therefore, $(E_\alpha \cap \omega) \setminus P \neq \emptyset$ or there is $A \in E_\alpha \subseteq E$ such that $|A \setminus P| = \aleph_0$. This completes the proof. ■

4 Non-pseudocompact $2^{\Psi(\mathcal{A})}$

In this section we will provide a consistent example of a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is not pseudocompact.

Recall that $\mathcal{D} \subseteq [\omega]^\omega$ is *dense* if for every $B \in [\omega]^\omega$ there is $D \in \mathcal{D}$ such that $D \subseteq^* B$. The *distributivity number* $\mathfrak{h}$ of $[\omega]^\omega$ is defined as the minimal size of a collection of dense downward closed subsets of $[\omega]^\omega$ whose intersection is empty. It is well known that $\mathfrak{p} \le \mathfrak{h} \le \mathfrak{c}$ and that both inequalities are consistently strict.

Theorem 4.1 ([BPS80]) *There is a family $\mathcal{T} \subseteq [\omega]^\omega$ such that*

(1) $\mathcal{T}$ is a tree (ordered by $\supseteq^*$) of height $\mathfrak{h}$.
(2) Each level of $\mathcal{T}$ is a maximal antichain in $[\omega]^\omega$ (a MAD family).
(3) Each $D \in \mathcal{T}$ has $\mathfrak{c}$-many immediate successors.
(4) $\mathcal{T}$ is a dense subset of $[\omega]^\omega$.

This is the *base tree theorem* of B. Balcar, J. Pelant and P. Simon.

Theorem 4.2 ($\mathfrak{h} < \mathfrak{c}$) *There is a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is not pseudocompact.*

Proof. Fix a base tree $\mathcal{T}$ of height $\mathfrak{h}$ as in Theorem 4.1. For $A \subseteq 2^{<\omega}$ let $\pi_A = \{n \in \omega : A \cap 2^n \neq \emptyset\}$. Let $\mathcal{A} \subseteq [2^{<\omega}]^\omega$ be such that

(1) $\mathcal{A}$ is a MAD family (of subsets of $2^{<\omega}$),
(2) every $A \in \mathcal{A}$ is either a chain or an antichain in $2^{<\omega}$,
(3) $\pi_A \in \mathcal{T}$ for all $A \in \mathcal{A}$,
(4) $A, B \in \mathcal{A}$ and $A \neq B$ implies $\pi_A \neq \pi_B$.

Such a MAD family $\mathcal{A}$ exists by a simple application of Zorn-Kuratowski Lemma.

To show that $2^{\Psi(\mathcal{A})}$ is not pseudocompact let $Y = \{F_m : m \in \omega\}$, where $F_m = 2^m$, the set of all binary sequences of length m. We will show that Y has no accumulation point in $2^{\Psi(\mathcal{A})}$. Notice that an accumulation point F of Y, if there is any, must be contained in $\mathcal{A}$, for if $s \in F \cap 2^{<\omega}$, then $U = \langle \Psi(\mathcal{A}); \{s\} \rangle$ is a neighbourhood of F for which $|U \cap Y| \leq 1$. To see that there are no accumulation points, let $F \subseteq \mathcal{A}$. Then there are two cases:

Case 1: $|F| < \mathfrak{c}$.

There is an $f \in 2^\omega$ such that $B_f = \{f \restriction n : n \in \omega\}$ has finite intersection with all members A of F. Then

$$U = \left\{H \in \Psi(\mathcal{A}) : H \cap \mathrm{cl}_{\Psi(\mathcal{A})}(B_f) = \emptyset\right\} = \left\langle \Psi(\mathcal{A}) \setminus \mathrm{cl}_{\Psi(\mathcal{A})}(B_f) \right\rangle$$

is a neighbourhood of F, which contains no F_n.

Case 2: $|F| = \mathfrak{c} > \mathfrak{h}$.

By (3) and (4), the set $\{\pi_A : A \in F\} \subseteq \mathcal{T}$ is not a branch of the base tree $\mathcal{T}$. So, there are $A, B \in F$ such that $\pi_A \cap \pi_B \subseteq k$ for some $k \in \omega$. Then $W = \langle \Psi(\mathcal{A}); A \backslash k, B \backslash k \rangle$ is neighbourhood of F, yet $W \cap Y = \emptyset$. So, F is not an accumulation point of Y. ■

The assumption $\mathfrak{h} < \mathfrak{c}$ can be weakened to the existence of a base tree without branches of length $\mathfrak{c}$. We conjecture an affirmative answer to the following question:

Problem 4.3 *Is there, in ZFC, a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is pseudocompact?*

5 ZFC example

Ginsburg's Theorem 1.1(a) says that 2^X is countably compact whenever every power of the space X is countably compact. It is a known and easy to prove fact that, for a topological space X, the space X^ω is pseudocompact if and only if X^κ is pseudocompact for every cardinal κ. In this section we construct a space X (subspace of $\beta\omega$ — the Čech-Stone compactification of ω) such that X^ω is pseudocompact, yet 2^X is not. Therefore, a result analogous to Ginsburg's does not hold for pseudocompactness.

Given an ordinal number α, we denote the set of limit ordinals below α by $\lim(\alpha)$. As usual, we identify $\beta\omega$ with the set all ultrafilters on ω and, in particular, $\omega^* = \beta\omega\backslash\omega$ with the set of all free ultrafilters on ω. For a set $A \subseteq \omega$, let $A^* = \{p \in \omega^* : A \in p\}$ and $\overline{A} = A \cup A^*$. Recall that given a topological space X, an ultrafilter $q \in \omega^*$, a point $x \in X$ and a sequence of points $\langle x_n : n \in \omega\rangle \subseteq X$, we say that x is a *q-limit* of the sequence $\langle x_n : n \in \omega\rangle$, $x = q\text{-}\lim\langle x_n : n \in \omega\rangle$, if for every neighbourhood U of x the set $\{m \in \omega : x_m \in U\}$ is an element of q.

Theorem 5.1 *There is a subspace X of $\beta\omega$ such that X^ω is pseudocompact yet 2^X is not.*

Proof. Enumerate all sequences of elements of ω^ω by $\{f_\alpha : \alpha \in \lim(\mathfrak{c})\}$, where each $f_\alpha = \langle f_{\alpha,n} : n \in \omega\rangle$. Let $Y = \langle F_n : n \in \omega\rangle$, where $F_n = [2^n, 2^{n+1})$. Given $U \subseteq \omega$, let $\pi_U = \{n \in \omega : U \cap F_n \neq \emptyset\}$, and for an ultrafilter p, let $\pi(p) = \{\pi_U : U \in p\}$, and observe that $\pi(p)$ is an ultrafilter as well.

To carry out the construction we will choose, for $\alpha \in \lim(\mathfrak{c})$, an ultrafilter $q_\alpha \in \omega^*$ and a set $X_\alpha = \{p_{\alpha+m} : m \in \omega\} \subseteq \beta\omega$ so that, for every $\alpha \in \lim(\mathfrak{c})$ and $m \in \omega$:

(1) $p_{\alpha+m} = q_\alpha\text{-}\lim\langle f_{\alpha,n}(m) : n \in \omega\rangle$,
(2) there is $U \in p_{\alpha+m}$ such that U is a partial selector of $\{F_k : k \in \omega\}$, i.e. $|F_k \cap U| \leq 1$ for each $k \in \omega$,
(3) for every $\beta < \alpha$, there is $U \in p_{\alpha+m}$ and $V \in p_\beta$ such that $\pi_U \cap \pi_V = \emptyset$.

Assume that this can be accomplished and let $X = \omega \cup \bigcup\{X_\alpha : \alpha < \lim(\mathfrak{c})\}$. To show that 2^X is not pseudocompact it suffices to prove that Y has no accumulation point in 2^X. Aiming towards a contradiction, assume that $F \in 2^X$ is an accumulation point of Y.

Note that $F \cap \omega = \emptyset$. If $F \cap \omega \neq \emptyset$ let $m \in F \cap \omega$. There is a unique $k \in \omega$ such that $m \in F_k$. Put $W = \langle X; \{m\}\rangle$. Then W is a neighbourhood of F and $W \cap Y = \{F_k\}$. Thus F cannot be an accumulation point.

CASE 1: F is countable.

By property (2) above, for every $p \in F$ there is $U_p \in p$ such that U_p is a partial selector of Y. Choose $K = \{x_m : m \in \omega\}$ such that $x_m \in F_m$ for every $m \in \omega$ and $|U_p \cap K| < \omega$ for every $p \in F$. Put

$$W = \left\{H \in 2^X : H \cap \overline{K} = \emptyset\right\} = \left\langle X \backslash \overline{K} \right\rangle.$$

Notice that $|U_p \cap K| < \omega$ implies $p \notin \overline{K}$ and thus W is a neighbourhood of F. Moreover, $F_m \notin W$ as $x_m \in \overline{K} \cap F_m$ for every $m \in \omega$. Hence $W \cap Y = \emptyset$, contradiction.

CASE 2: F is uncountable.

Property (3) above ensures the existence of $p, q \in F$, $U \in p$ and $V \in q$ such that $\pi_U \cap \pi_V = \emptyset$. Let $W = (X; \overline{U} \cap X, \overline{V} \cap X)$. Then W is a neighbourhood of F. But, $F_k \notin W$ for every $k \in \omega$, contradiction.

To show that X^ω is pseudocompact, let $\langle h_n : n \in \omega\rangle$ be a sequence of elements of ω^ω. There exists $\alpha \in \omega$ such that $f_\alpha = \langle h_n : n \in \omega\rangle$. Define $h \in X^\omega$ by $h(m) = q_\alpha\text{-}\lim f_{\alpha,n}(m)$. Clause (1) assures that h is the q_α-limit of $\langle h_n : n \in \omega\rangle$. Thus X^ω is pseudocompact by Proposition 2.3.

To conclude the proof of the theorem, we show how to carry out the recursive construction satisfying the properties (1) to (3). Suppose we are at stage α and that the sets X_β and the ultrafilters q_β have been chosen for $\beta \in \lim(\alpha)$. Let $g_m(n) = f_{\alpha,n}(m)$, for every $m, n \in \omega$.

Claim 1 *There exists $C \in [\omega]^\omega$ such that:*

(a) *For every $m \in \omega$ there is a $k \in \omega$ such that $g_m \restriction (C\backslash k)$ is constant or $g_m \restriction (C\backslash k)$ is injective, and $g_m[C\backslash k]$ is a partial selector of Y,*
(b) *for every $m \neq n \in \omega$, $g_m \restriction C =^* g_n \restriction C$ or $g_m[C] \cap g_n[C] =^* \emptyset$,*
(c) *for every $\beta < \alpha$ and every $m \in \omega$ such that $g_m \restriction C$ is not eventually constant, $\pi_{g_m[C]} \cap \pi_V = \emptyset$ for some $V \in p_\beta$.*

In order to prove the claim, let

$$N = \{\pi(p) \in \omega^* : (\exists \beta < \alpha)\,(p \in X_\beta)\}$$

and recursively construct a decreasing sequence $\{A_m : m \in \omega\} \subseteq [\omega]^\omega$ such that for every $m \in \omega$:

(1) $g_m \restriction A_m$ is constant or $g_m \restriction A_m$ is injective, and $g_m[A_m]$ is a partial selector of Y,

(2) for every $k < m \in \omega$, $g_m \restriction A_m =^* g_k \restriction A_m$ or $g_m[A_m] \cap g_k[A_m] =^* \emptyset$, and
(3) $(\pi_{g_m[A_m]})^* \cap N = \emptyset$.

Put $A_{-1} = \omega$. To carry out the construction assume that $\{A_k : k < m\}$ have been defined and consider the function g_m. It is easy to find B an infinite subset of A_{m-1} such that (1) $g_m \restriction B$ is constant or $g_m \restriction B$ is injective, $g_m[B]$ is a partial selector of Y and (2) for every $k < m \in \omega$, $g_m \restriction B =^* g_k \restriction B$ or $g_m[B] \cap g_k[B] =^* \emptyset$. Since $|N| < \mathfrak{c}$, N is nowhere dense in $\beta\omega$ (This follows from the existence of AD families of size $\mathfrak{c}$, see [vM84]). Thus there is an infinite $D \subseteq g_m[B]$ such that $(\pi_D)^* \cap N = \emptyset$. Let $A_m = g_m^{-1}[D]$. Conditions (1) - (3) are obviously satisfied.

To conclude the proof of the claim, choose $C \in [\omega]^\omega$ such that $C \subseteq^* A_m$ for every $m \in \omega$. The set C clearly satisfies the first two clauses. To see that it also satisfies the third, let $\beta < \alpha$ and let $m \in \omega$. Since $C \subseteq^* A_m$, then $\pi(p_\beta) \notin \left(\pi_{g_m[C]}\right)^*$ and hence there is $V \in p_\beta$ such that $\pi_V \cap \pi_{g_m[C]} = \emptyset$.

Choose $q_\alpha \in (C)^*$ and let $p_{\alpha+m} = q_\alpha\text{-}\lim \langle g_m(n) : n \in \omega \rangle \in (g_m[C])^*$ for every $m \in \omega$. Then q_α and $X_\alpha = \{p_{\alpha+m} : m \in \omega\}$ satisfy the properties (1), (2) and (3). Indeed, the fact that (1) and (2) hold follows directly from the construction. To check (3), let $\beta < \alpha$ and $m \in \omega$. By (c) there is a $V \in p_\beta$ such that $\pi_U \cap \pi_V = \emptyset$, where $U = g_m[C] \in p_{\alpha+m}$. ■

References

[Bel81] Murray G. Bell, *On the combinatorial principle $P(\mathfrak{c})$*, Fund. Math. **114** (1981), no. 2, 149–157.

[BJ95] Tomek Bartoszyński and Haim Judah, *Set theory: On the structure of the real line*, A K Peters Ltd., Wellesley, MA, 1995.

[BPS80] Bohuslav Balcar, Jan Pelant, and Petr Simon, *The space of ultrafilters on* **N** *covered by nowhere dense sets*, Fund. Math. **110** (1980), no. 1, 11–24.

[CNT04] Jiling Cao, Tsugunori Nogura, and A. H. Tomita, *Countable compactness of hyperspaces and Ginsburg's questions*, Topology Appl. **144** (2004), no. 1-3, 133–145.

[Eng89] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from Polish by the author.

[Gin75] John Ginsburg, *Some results on the countable compactness and pseudocompactness of hyperspaces*, Canad. J. Math. **27** (1975), no. 6, 1392–1399.

[Jec97] Thomas Jech, *Set theory*, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1997.

[Kun83] Kenneth Kunen, *Set theory: An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1983, Reprint of the 1980 original.

[vD84] Eric K. van Douwen, *The integers and topology*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 111–167.

[vM84] Jan van Mill, *An introduction to $\beta\omega$*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 503–567.