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Topology and its Applications 154 (2007) 2081–2088
Topology and its Applications
www.elsevier.com/locate/topol

Supremum vs. maximum: λ-sets

Fernando Hernández-Hernández [1]

Instituto de Matemáticas, UNAM (Morelia), Apartado postal 61-3 (Xangari), Morelia Michoacán, Mexico 58089

Received 20 September 2005; received in revised form 4 July 2006; accepted 30 July 2006

Abstract

We show that, relative to the existence of an inaccessible cardinal, it is consistent that there is no λ-set of maximal size and that in the absence of inaccessible cardinals there is a λ-set of maximal size.

MSC: 03E35; 54A35

Keywords: λ-sets; Special subsets of reals; Strongly inaccessible cardinal; Hechler forcing; Isomorphism of names

1. Introduction

K. Kuratowski introduced *rarefied* or λ-sets in 1933. A set $X \subseteq \mathbb{R}$ is a *λ-set* if every countable subset of X is relatively G_δ in X. N. Lusin showed that λ-sets of cardinality $\aleph_1$ always exist and F. Rothberger improved Lusin's result by showing that ZFC suffices to show that there is a λ-set of cardinality $\mathfrak{b}$ (see Theorem 2.4).

The motivation for this paper comes from Problem 291 in [8] which asks about pseudonormal Ψ-spaces. Recall that a space is *pseudonormal* if every pair of disjoint closed sets can be separated by disjoint open sets, provided at least one of them is countable. An *almost disjoint family* $\mathcal{A}$ is a family of infinite subsets of ω (or any other countable set) such that $A \cap B$ is finite for distinct $A, B \in \mathcal{A}$. The *Ψ-space* $\Psi(\mathcal{A})$ associated to $\mathcal{A}$ is $\omega \cup \mathcal{A}$ where the points of ω are isolated and the basic neighbourhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus F$, where $F \subseteq \omega$ is finite. The space $\Psi(\mathcal{A})$ is always a first countable, separable, locally compact Moore space. It is known that if $\Psi(\mathcal{A})$ is a pseudonormal space, then $\mathcal{A}$ is a λ-set as subspace of $\mathcal{P}(\omega)$, and if there is a λ-set of cardinality κ then there is an almost disjoint family $\mathcal{A}$ of cardinality κ such that $\Psi(\mathcal{A})$ is a pseudonormal space (see [4, Proposition 2.2]). F.B. Jones used λ-sets to construct a pseudonormal Moore space which served as the inspiration for nonmetrizable normal Moore spaces based on Q-sets. For more on λ-sets consult [6,7].

Problem 291 in [8] has two parts; in the first one P. Nyikos asked: Is there a pseudonormal Ψ-space of cardinality $\mathfrak{d}$? The first part of Problem 291 is equivalent to asking whether there is a λ-set of size $\mathfrak{d}$. A.W. Miller [7, Theorem 22] showed that in the Cohen model (in which $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \aleph_2$), any λ-set of reals has size $\aleph_1$. Thus in the Cohen model the answer is "no". Of course, in a model of $\mathfrak{b} = \mathfrak{d}$ the answer is "yes", see Theorem 2.4.

E-mail address: fernando@matmor.unam.mx.
[1] The author gratefully acknowledges partial research support from PAPIIT Grant IN106705.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.07.016

Problem 291 has second part where Nyikos asked: More generally, what is the maximum cardinality of a pseudo-normal Ψ-space? This note is meant to show that the second part of Problem 291 is in general ill-posed. The results in this paper were announced in [4]. We show that: (1) In the absence of inaccessible cardinals, there is a λ-set of maximal cardinality and (2) assuming the existence of a strongly inaccessible cardinal it is consistent that: $\mathfrak{c}$ is a limit cardinal and for every $\kappa < \mathfrak{c}$ there is a λ-set of size κ, yet there is no λ-set of size $\mathfrak{c}$. That is, the maximum cardinality of a λ-set may not be attained.

Our terminology is mostly standard: For functions $f, g \in \omega^\omega$ we write $f \leqslant^* g$ to mean that there is some $m \in \omega$ such that $f(n) \leqslant g(n)$ for all $n \geqslant m$. The bounding number in ω^ω, $\mathfrak{b}$, is the least cardinal of an $\leqslant^*$-unbounded family of functions. The dominating number in ω^ω, $\mathfrak{d}$, is the least cardinal of a $\leqslant^*$-cofinal family of functions. The set ω^ω is equipped with the product topology, that is the topology with basic open sets of the form $[s] = \{f \in \omega^\omega\colon\ s \subseteq f\}$, where $s \in \omega^{<\omega}$. See [5] for undefined set theoretical notions.

2. The results

The first result we are presenting shows that regarding the cardinalities of λ-sets, sup = max if there are no inaccessible cardinals below the continuum. In other words, the maximum of the cardinalities of λ-sets is attained, this shows that the second part of Problem 291 makes sense in the absence of inaccessible cardinals.

Theorem 2.1. *If κ is a singular cardinal and there is a λ-set of cardinality μ for every cardinal $\mu < \kappa$, then there is a λ-set of cardinality κ.*

Proof. Let $\mathrm{cf}(\kappa) = \mu$ and consider an increasing sequence $\langle \kappa_\alpha\colon\ \alpha < \mu\rangle$ which is cofinal in κ. Choose a λ-set $A \subseteq \mathbb{R}$ of cardinality μ and for each $\alpha < \mu$ choose a λ-set $X_\alpha \subseteq \mathbb{R}$ of cardinality κ_α. Enumerate A as $\{x_\alpha\colon\ \alpha < \mu\}$. We claim that $X = \bigcup_{\alpha<\mu}\{x_\alpha\} \times X_\alpha$ is a λ-set of cardinality κ. That $|X| = \kappa$ is clear. If $Y \in [X]^{\aleph_0}$ then there is some $A_0 = \{x_{\alpha_n}\colon\ n \in \omega\} \subseteq A$ such that Y is contained in $\bigcup_{n\in\omega}\{x_{\alpha_n}\} \times X_{\alpha_n}$ and moreover $Y_n = Y \cap (\{x_{\alpha_n}\} \times X_{\alpha_n})$ is at most countable for every $n \in \omega$. Since $\{x_{\alpha_n}\colon\ n \in \omega\}$ is a G_δ subset of A, there are open sets B_m, for $m \in \omega$, such that $\bigcup_{n\in\omega}\{x_{\alpha_n}\} \times X_{\alpha_n} = \bigcap_{m\in\omega} B_m$.

On the other hand, Y_m is a G_δ subset of $\{x_{\alpha_m}\} \times X_{\alpha_m}$ and $\{x_{\alpha_m}\} \times X_{\alpha_m}$ is a G_δ subset of X as well, thus Y_m is a G_δ subset of X. Therefore we can fix open subsets $U_{m,n}$ of X such that

(1) $U_{m,n+1} \subseteq U_{m,n}$ for all $n \in \omega$,
(2) $U_{m,n} \cap (\{x_{\alpha_k}\} \times X_{\alpha_k}) = \emptyset$ for all $k < m$, and
(3) $Y_m = \bigcap_{n\in\omega} U_{m,n}$.

Let $W_n = (\bigcup_{m\in\omega} U_{m,n}) \cap B_n$, for $n \in \omega$. Then W_n is open subset of X and it can be easily verified that $Y = \bigcap_{n\in\omega} W_n$. □

Corollary 2.2. *Let $\Lambda = \sup\{|X|\colon\ X$ is a λ-set$\}$. If Λ is not an inaccessible cardinal then there is a λ-set X with $|X| = \Lambda$.*

Now we turn to address the case where inaccessible cardinals exist. Here we see that it is consistent that the maximum of the cardinalities of λ-sets is never attained. It will be easier to change the real numbers $\mathbb{R}$ by ω^ω and show the result for subsets of ω^ω. Since every λ-set is zero-dimensional, there is no difference to consider λ-sets in the real line or in the irrationals.

Theorem 2.3. *If there is a strongly inaccessible cardinal κ then there is a model of* ZFC *in which there is a λ-set of every cardinality below $\mathfrak{c}$, yet there is no λ-set of size $\mathfrak{c}$.*

The main idea to prove the theorem is to make the continuum a limit cardinal and to add a λ-set of each cardinal below $\mathfrak{c}$. The natural forcing to add a λ-set of cardinality μ is the finite support iteration $\mathbb{P}_\mu$ of length μ of Hechler forcing $\mathbb{D}$. This is due to the fact that $\mathbb{P}_\mu$ adds a well-ordered family of functions in ω^ω of size μ, the dominating (or Hechler) reals. Thus $\mathbb{P}_\mu$ forces $\mathfrak{b} \geqslant \mu$. Then the existence of a λ-set of cardinality μ follows from the following theorem. For a proof of it see [7].

Theorem 2.4 *(Rothberger). There exists a λ-set of cardinality* $\mathfrak{b}$.

A second key ingredient of the idea is the following simple lemma. We give a proof for the sake of completeness.

Lemma 2.5 *(Folklore). If X is a λ-set and $\mathbb{P}$ is a c.c.c. forcing notion then $\mathbb{P}$ preserves the fact that X is a λ-set.*

Proof. Fix X and $\mathbb{P}$ as in the statement of the lemma. It is well known that the cardinality of X is preserved. If $\dot{A}$ is a $\mathbb{P}$-name for a countable subset of X, then there is a countable $B \subseteq X$ in the ground model such that $\mathbb{P} \Vdash \dot{A} \subseteq B$. Since X is assumed a λ-set in the ground model, there are open sets $U_n \subseteq X$, $n \in \omega$, such that $U_n = B$. Then working in the extension $\mathbf{V}[G]$, where G is a $\mathbb{P}$-generic filter over $\mathbf{V}$, we can fix an enumeration $\{b_n \colon n \in \omega\}$ of $B \setminus A$ and thus write $A = \bigcap_{n\in\omega} U_n \setminus \{b_n\}$. Therefore A is a G_δ subset of X, in $\mathbf{V}[G]$. □

Thus we can use iterations of Hechler forcing to add λ-sets of every cardinality below the new $\mathfrak{c}$ and we can preserve those λ-sets if we use a c.c.c. forcing notion which puts together (in an independent manner) all the iterations that we are going to use. The forcing notion $\mathbb{P}$ that we will use is defined next. Before doing that, recall that conditions in Hechler forcing $\mathbb{D}$ are of the form $\langle s, f\rangle$ where $s \in \omega^{<\omega}$ and $f \in \omega^\omega$, and the ordering is $\langle s, f\rangle \leqslant \langle t, g\rangle$ if and only if

(1) $s \supseteq t$,
(2) $f \geqslant g$ and
(3) $(\forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t))\ (s(n) \geqslant g(n))$.

Definition 2.6. Let κ be a strongly inaccessible cardinal and let $\{\mu_\alpha \colon \alpha < \kappa\}$ be a fix enumeration of all regular cardinals below κ. Then define $\mathbb{P}$ by $p \in \mathbb{P}$ if and only if $p \in \prod_{\alpha<\kappa} \mathbb{P}_{\mu_\alpha}$ and $\mathrm{supp}(p)$ is finite, where $\mathbb{P}_{\mu_\alpha}$ is a finite support iteration of length μ_α of Hechler forcing $\mathbb{D}$.

Proof of Theorem 2.3. We use the forcing $\mathbb{P}$ just described above. It is clear that $\mathbb{P} \Vdash \mathfrak{c} = \kappa$. That $\mathbb{P}$ is a c.c.c. forcing notion follows from the facts that finite support iteration of forcing notions with precaliber[2] $\aleph_1$ has precaliber $\aleph_1$ and that a finite support product of forcing notions with precaliber $\aleph_1$ has precaliber $\aleph_1$ (see [1, Chapter 1]).

On the other hand, to show that there is a λ-set of cardinality μ for every cardinal $\mu < \kappa$, fix an arbitrary $\mathbb{P}$-generic filter G over $\mathbf{V}$. As $\mathbb{P}$ is a product of forcing notions, G can be seen as the product of G_α's where G_α is $\mathbb{P}_{\mu_\alpha}$-generic over $\mathbf{V}$. Moreover, given any cardinal $\mu < \kappa$, there is some $\alpha < \kappa$ such that $\mu < \mu_\alpha$ and

$$\mathbf{V}[G] = \mathbf{V}\Big[\prod_{\alpha<\kappa} G_\alpha\Big] = \mathbf{V}[G_\alpha]\Big[\prod_{\beta\in\kappa\setminus\{\alpha\}} G_\beta\Big].$$

Since $\mathbb{P}_\alpha$ forces $\mathfrak{b} = \mu_\alpha$, by Theorem 2.4 it follows that there is a λ-set X of cardinality μ_α in $\mathbf{V}[G_\mu]$ and since the next part of the extension is given by a c.c.c. extension, Lemma 2.5 implies that X remains a λ-set in $\mathbf{V}[G]$. Then that there is a λ-set in $\mathbf{V}[G]$ of cardinality μ follows from the fact that subsets of λ-sets are λ-sets themselves.

It remains to show that there is no λ-set of cardinality $\mathfrak{c}$ in the generic extension $\mathbf{V}[G]$. This will follow by the next lemma. □

Lemma 2.7. *Let $\mathbb{P}$ be the forcing notion of Definition* 2.6 *and let $Y = \{\dot{f}_\xi \colon \xi < \kappa\}$ be a set of $\mathbb{P}$-names for reals which are forced by $\mathbb{P}$ to be pairwise distinct. If G is a $\mathbb{P}$-generic filter over* $\mathbf{V}$, *then $\{f_\xi \colon \xi < \kappa\}$ is not a λ-set in* $\mathbf{V}[G]$.

We will spend the next section to give a proof of this lemma in a detailed way. The idea for the proof is to take a countable elementary submodel M of some big enough $H(\vartheta)$ such that $\mathbb{P}, Y \in M$, then we shall show that whenever $\dot{W}$ is a $\mathbb{P}$-name for a G_δ subset of ω^ω, there are κ-many ξ's such that $\mathbb{P} \Vdash \dot{f}_\xi \in \dot{W}$. In order to do this we will need

[2] Recall that a forcing notion $\mathbb{P}$ has *precaliber κ* if for every subset $\mathcal{A}$ of size κ there is $\mathcal{B} \subseteq \mathcal{A}$ of size κ such that $\mathcal{B}$ is centred.

to introduce several notions among which a notion of similarity between names for functions in ω^ω will be crucial. There the fact that Hechler forcing is definable will also be very important.

3. Proof the main lemma

In this concluding section we are going to use an isomorphism of names type argument to give a proof of the main lemma and hence to complete the proof of Theorem 2.3. The use of arguments involving isomorphic names has been around in the literature for quite a long time without been explicitly mention. See for example [3]. At the beginning of Section 3 in [2] a particular argument of isomorphic names for Cohen forcing is explained in some detail. The main result in [4] also uses this type of argument.

A *nice* $\mathbb{P}$*-name* $\dot{f}$ for a function in ω^ω will be a $\mathbb{P}$-name of the form

$$\bigcup_{n\in\omega}\{\langle p,\langle n,i_n^p\rangle\rangle:\ p\in\mathcal{A}_n\ \&\ i_n^p\in\omega\},$$

where

(1) $\mathcal{A}_n$ is a maximal antichain in $\mathbb{P}$ for every $n\in\omega$, and
(2) $\langle p,\langle n,i_n^p\rangle\rangle\in\dot{f}\Longrightarrow p\Vdash\dot{f}(n)=i_n^p$.

It is easy (see [5, 5.12]) to show that whenever $\mathbb{P}\Vdash\dot{f}\in\omega^\omega$ there is a nice name $\dot{f}'$ such that $\mathbb{P}\Vdash\dot{f}=\dot{f}'$; that is, every function in the generic extension has a nice $\mathbb{P}$-name.

Now, let $\mathbb{P}_\alpha$ be a finite support iteration of length α of Hechler forcing $\mathbb{D}$. By induction on $\alpha\geqslant 1$ we define when p is a good condition in $\mathbb{P}_\alpha$. Firstly, for $\alpha=1$, every condition p in $\mathbb{D}$ is a good condition. Suppose we have already defined good conditions in $\mathbb{P}_\alpha$. If $p\in\mathbb{P}_\alpha*\dot{\mathbb{D}}$, then p is a *good condition* if $p\restriction\alpha$ is a good condition in $\mathbb{P}_\alpha$ and

$$p\restriction\alpha\Vdash p(\alpha)=\langle s,\dot{f}\rangle,$$

where $s\in\omega^{<\omega}$ and $\dot{f}$ is a nice $\mathbb{P}_\alpha$-name for a function in ω^ω such that conditions p appearing in the nice $\mathbb{P}_\alpha$-name $\dot{f}$ are all good conditions in $\mathbb{P}_\alpha$. If α is a limit ordinal and we have defined good conditions in $\mathbb{P}_\beta$ for all $\beta<\alpha$, then $p\in\mathbb{P}_\alpha$ is a *good condition* if $p\restriction\beta$ is a good condition in $\mathbb{P}_\beta$, where $\beta=\max(\operatorname{supp}(p))+1$.

Lemma 3.1. *For every $\alpha<\kappa$, $\mathbb{P}_\alpha$ has a dense subset of good conditions.*

Proof. We can prove this lemma by induction on α. Since $\mathbb{P}_\alpha$ is a finite support iteration, the limit steps of the induction are easy. To show the successor steps, assume $\mathbb{P}_\alpha$ has a dense subset of good conditions and let p be an arbitrary condition in $\mathbb{P}_\alpha*\dot{\mathbb{D}}$. Then $p\restriction\alpha\Vdash p(\alpha)=\langle\dot{s},\dot{f}\rangle\in\dot{\mathbb{D}}$ and using the density of good conditions in $\mathbb{P}_\alpha$ there is a good condition $\widetilde{p}\restriction\alpha\leqslant p\restriction\alpha$ and a nice $\mathbb{P}_\alpha$-name $\dot{f}'$ such that $\widetilde{p}\restriction\alpha\Vdash\dot{s}=s\in\omega^{<\omega}$, $\widetilde{p}\restriction\alpha\Vdash\dot{f}=\dot{f}'$ and every condition $q\in\mathbb{P}_\alpha$ which appears in $\dot{f}'$ is a good condition in $\mathbb{P}_\alpha$. Put $\widetilde{p}=(\widetilde{p}\restriction\alpha)\cup\{\langle\alpha,\langle s,\dot{f}'\rangle\rangle\}$. Then $\widetilde{p}$ is a good condition in $\mathbb{P}_\alpha*\dot{\mathbb{D}}$ and $\widetilde{p}\leqslant p$. $\square$

Now we extend the definition of good conditions to conditions in the whole forcing $\mathbb{P}$ in the natural way: A condition $p\in\mathbb{P}$ is a *good condition* if $p(\alpha)\in\mathbb{P}_{\mu_\alpha}$ is a good condition for every $\alpha<\kappa$. The next lemma follows directly from Lemma 3.1.

Lemma 3.2. $\mathbb{P}$ *has a dense subset of good conditions.*

As $\mathbb{P}$ is a c.c.c. forcing notion, every function from ω into ω in a generic extension by $\mathbb{P}$ has a nice $\mathbb{P}$-name formed by countable antichains of good conditions. If $\dot{f}$ is a $\mathbb{P}$-name for an element of ω^ω, we say that $\dot{f}$ is a *good* $\mathbb{P}$*-name* for a function in ω^ω if $\dot{f}$ is a nice name and $\langle p,\langle n,k\rangle\rangle\in\dot{f}$ implies that p is a good condition in $\mathbb{P}$.

Again by induction we define the transitive support of good conditions in $\mathbb{P}_\alpha$. For $p\in\mathbb{D}$ we define the *transitive support* of p to be $\operatorname{tr\,supp}(p)=\{0\}$. Suppose we have defined the transitive support for good conditions in $\mathbb{P}_\alpha$, then for a good condition $p\in\mathbb{P}_{\alpha+1}$, the *transitive support* of p is $\operatorname{tr\,supp}(p)=\operatorname{tr\,supp}(p\restriction\alpha)$ if $p\restriction\alpha\Vdash p(\alpha)=1_{\mathbb{D}}$, and

$$\operatorname{tr\,supp}(p)=\operatorname{tr\,supp}(p\restriction\alpha)\cup\{\alpha\}\cup\operatorname{tr\,supp}(\dot{f})$$

if $p \restriction \alpha \not\Vdash p(\alpha) = 1_{\mathbb{D}}$ and $p \restriction \alpha \Vdash p(\alpha) = \langle s, \dot{f} \rangle$, where $\operatorname{tr\,supp}(\dot{f}) = \bigcup\{\operatorname{tr\,supp}(q)\colon q \in \mathcal{A}_n\}$ with $\langle \mathcal{A}_n : n \in \omega \rangle$ being the sequence of maximal antichains in $\mathbb{P}_\alpha$ used to define the good $\mathbb{P}_\alpha$-name $\dot{f}$. If α is a limit ordinal and we have defined the transitive support for good conditions in $\mathbb{P}_\beta$ for all $\beta < \alpha$, then for a good condition $p \in \mathbb{P}_\alpha$ we define the *transitive support* of p by

$$\operatorname{tr\,supp}(p) = \operatorname{tr\,supp}(p \restriction \beta),$$

where $\beta = \max(\operatorname{supp}(p)) + 1$.

For technical reasons, we assume, without loss of generality, that the transitive support of every good condition contains 0.

The motivation to introduce the transitive support for conditions is to define isomorphic good $\mathbb{P}$-names for functions in ω^ω. Before doing that, we need to extend the notion of transitive support for conditions in the whole forcing $\mathbb{P}$. This is easy as we can do it coordinatewise: If $p \in \mathbb{P}$ is a good condition then the *transitive support* of p is given by

$$\operatorname{tr\,supp}(p) = \bigcup\bigl\{\{\mu_\alpha\} \times \operatorname{tr\,supp}\bigl(p(\alpha)\bigr)\colon \alpha \in \operatorname{supp}(p)\bigr\}.$$

It is worth observing that $\operatorname{tr\,supp}(p)$ is a countable subset of $\kappa \times \kappa$ and it is meant to give us a template of the parts of our forcing used to form a good condition. The definition of the transitive support of a good $\mathbb{P}$-name for a function in ω^ω has implicitly been given and it might be thought as a template or guide in making that good $\mathbb{P}$-name accurately over the whole forcing $\mathbb{P}$. If $\dot{f}$ is a good $\mathbb{P}$-name for a function in ω^ω, then the *support of* $\dot{f}$ is defined by

$$\operatorname{supp}(\dot{f}) = \bigcup\bigl\{\operatorname{supp}(p)\colon (\exists n, k \in \omega)\bigl(\bigl\langle p, \langle n, k\rangle\bigr\rangle \in \dot{f}\bigr)\bigr\}.$$

The *transitive support of* $\dot{f}$ is simply

$$\operatorname{tr\,supp}(\dot{f}) = \bigcup\bigl\{\{\mu_\alpha\} \times \operatorname{tr\,supp}\bigl(p(\alpha)\bigr)\colon \alpha \in \operatorname{supp}(\dot{f})\bigr\}.$$

Again by c.c.c.-ness of $\mathbb{P}$ $\operatorname{supp}(\dot{f})$ is a countable subset of κ for every good $\mathbb{P}$-name for a function from ω into ω and $\operatorname{tr\,supp}(\dot{f})$ is a countable subset of $\kappa \times \kappa$.

The definition of an equivalence relation between good $\mathbb{P}$-names for functions will be given in terms of equivalence between templates. First we detail the notion of isomorphic templates.[3]

Definition 3.3. (1) A simple template is a countable subset of κ which (for technical reasons) contains 0.

(2) Two simple templates A and B are isomorphic if there is an order preserving bijection $\psi : A \to B$ such that:

(2a) if $\beta \in A$ and for some $\gamma \in \omega_1$, the next point of A is $\beta + \gamma$, then $\psi(\beta + \gamma) = \psi(\beta) + \gamma$, and
(2b) if $\beta \in A$ and the next point γ of A is such that $\beta + \omega_1 \leqslant \gamma$, then $\psi(\beta) + \omega_1 \leqslant \psi(\gamma)$.

(3) A template T is a subset of $\kappa \times \kappa$ of the form $T = \bigcup\{\{\alpha\} \times A_\alpha\colon \alpha \in S\}$ for an at most countable $S \subseteq \kappa$ and A_α being a simple template for every $\alpha \in S$.

(4) Two templates $T = \bigcup\{\{\alpha\} \times A_\alpha\colon \alpha \in S\}$ and $T' = \bigcup\{\{\alpha\} \times A'_\alpha\colon \alpha \in S'\}$ are isomorphic if there exists a bijection $\varphi : S \to S'$ such that:

(4a) $\varphi(\alpha) = \alpha$ for all $\alpha \in S \cap S'$,
(4b) for all $\alpha \in S$, the simple templates A_α and $A_{\varphi(\alpha)}$ are isomorphic witnessed by a bijection $\psi_\alpha : A_\alpha \to A'_{\varphi(\alpha)}$, and
(4c) in case $\varphi(\alpha) = \alpha$, then $A_\alpha = A_{\varphi(\alpha)}$ and ψ_α is the identity function.

(5) An isomorphism Φ between templates T and T' is a bijection $\Phi : T \to T'$ which induces bijections $\varphi : S \to S'$ and $\psi_\alpha : A_\alpha \to A_{\varphi(\alpha)}$ which witness that T and T' are isomorphic templates. We denote $\Phi(\alpha, \beta) = \langle \varphi(\alpha), \psi_\alpha(\beta) \rangle$ for $\langle \alpha, \beta \rangle \in T$.

It is easy to see that this notion of isomorphism between templates defines an equivalence relation on the family of templates and that every template is isomorphic to some template contained in $\omega \times \omega_2$ and, therefore, there are only

[3] Here the word "template" has no relationship with the same word used in the theory of iterations along a template.

$\aleph_2$-many distinct types of templates. Now we will use isomorphism of templates to translate information between names. Note that if $\dot{f}$ is a good $\mathbb{P}$-name for a function in ω^ω then $\operatorname{tr\,supp}(\dot{f})$ is a template. We need to make precise what we understand by translating information between conditions.

Let $\psi : A \to B$ by an order-preserving function between simple templates A and B. For conditions $p \in \mathbb{P}_\beta$ with $\operatorname{tr\,supp}(p) \subseteq A$, define a new condition $\overline{\psi}(p)$ by induction on $\beta \geqslant 1$. If $p \in \mathbb{P}_1$, then $\overline{\psi}(p)(0) = p(0)$. If we have defined $\overline{\psi}$ for all good conditions in $\mathbb{P}_\beta$ and $p \in \mathbb{P}_{\beta+1}$ is a good condition then:

(1) If $\beta \notin A$, then $\overline{\psi}(p) = \overline{\psi}(p \upharpoonright \beta)$, or
(2) if $\beta \in A$ and $p \upharpoonright \beta \Vdash p(\beta) = \langle s, \dot{f}\rangle$, then $\overline{\psi}(p) \upharpoonright \psi(\beta) = \overline{\psi}(p \upharpoonright \beta)$ and $\overline{\psi}(p) \upharpoonright \psi(\beta) \Vdash \overline{\psi}(p)(\psi(\beta)) = \langle s, \overline{\psi}(\dot{f})\rangle$, where

$$\overline{\psi}(\dot{f}) = \bigcup_{n\in\omega} \{\langle \overline{\psi}(q), \langle n, k_n^q\rangle\rangle : q \in \mathcal{A}_n \ \&\ k_n^q \in \omega\}$$

if $\dot{f} = \bigcup_{n\in\omega}\{\langle q, \langle n, k_n^q\rangle\rangle : q \in \mathcal{A}_n \ \&\ k_n^q \in \omega\}$.

And if $\beta \in A$ is a limit ordinal, we have defined $\overline{\psi}$ for all good conditions in $\mathbb{P}_\gamma$, for all $\gamma < \beta$, and $p \in \mathbb{P}_\beta$ is a good condition with $\gamma_0 = \max(\operatorname{supp}(p)) \in A$ then

$$\overline{\psi}(p) = \overline{\psi}\big(p \upharpoonright \psi(\gamma_0)\big) \cup \big\{\langle \gamma, 1_{\dot{\mathbb{D}}}\rangle : \psi(\gamma_0) \leqslant \gamma < \psi(\beta)\big\}.$$

It is not hard to check by induction that for every good condition $p \in \mathbb{P}_\beta$ such that $\operatorname{tr\,supp}(p) \subseteq A$ the following equality holds

$$\psi\big[\operatorname{tr\,supp}(p)\big] = \operatorname{tr\,supp}\big(\overline{\psi}(p)\big).$$

Thus if $p \in \mathbb{P}_\beta$ is a good condition and ψ is an isomorphism between $\operatorname{tr\,supp}(p)$ and some simple template A, then $A = \operatorname{tr\,supp}(\overline{\psi}(p))$.

Definition 3.4. Let $\Phi : T \to T$ be an isomorphism between templates T and T' where $\Phi(\alpha, \beta) = (\varphi(\alpha), \psi_\alpha(\beta))$ for all $\langle \alpha, \beta\rangle \in T$. Let $p \in \mathbb{P}$ be a good condition with $\operatorname{tr\,supp}(p)$ contained in T. The translated condition $\Phi(p)$ is defined by $\Phi(p)(\alpha) = \overline{\psi_\alpha}(p(\alpha))$ for all $\alpha < \kappa$ such that $\langle \alpha, 0\rangle \in T$ and $\Phi(p)(\alpha) = 1_{\mathbb{P}_{\mu_\alpha}}$ otherwise. Where $\overline{\psi_\alpha}$ is defined before, for every α.

The idea behind the previous definition is simple: Φ must translate p into $\Phi(p)$ using the templates as guides for making that translation accurate changing only the corresponding coordinates of conditions in the iterations $\mathbb{P}_{\mu_\alpha}$'s used to form the good $\mathbb{P}$-condition p. The function Φ can be thought as an isomorphism between the suborders of $\mathbb{P}$ induced by the respective templates T and T'. The isomorphism Φ cannot be a regular embedding though. To avoid too many indexes we will write $\psi_\alpha(p)$ instead of $\psi_\alpha(p(\alpha))$. Finally, the main definition of the section.

Definition 3.5. Let $\dot{f} = \bigcup_{n\in\omega}\{\langle p, \langle n, i\rangle\rangle : p \in \mathcal{A}_n\}$ and $\dot{g} = \bigcup_{n\in\omega}\{\langle q, \langle n, i\rangle\rangle : q \in \mathcal{B}_n\}$ be good $\mathbb{P}$-names for functions from ω into ω, we will say that they are isomorphic ($\dot{f} \approx \dot{g}$) if:

(1) The templates $\operatorname{tr\,supp}(\dot{f})$ and $\operatorname{tr\,supp}(\dot{g})$ are isomorphic witnessed by an isomorphism $\Phi : \operatorname{tr\,supp}(\dot{f}) \to \operatorname{tr\,supp}(\dot{g})$ and
(2) if $\langle p, \langle n, i\rangle\rangle \in \dot{f}$, $\langle q, \langle n, j\rangle\rangle \in \dot{g}$ and $q = \Phi(p)$, then $i = j$.

We are now ready to finish the proof of the main lemma.

Proof of Lemma 2.7. Suppose that $\{\dot{f}_\xi : \xi < \kappa\}$ are good $\mathbb{P}$-names for κ-many distinct reals. We will show that if G is a $\mathbb{P}$-generic filter over $\mathbf{V}$, then $X = \{f_\xi : \xi < \kappa\}$ is not a λ-set in $\mathbf{V}[G]$. Since subsets of λ-sets are λ-sets, to reach our goal we can thin out the original set in order to find a subset which is not a λ-set.

Because κ is a regular cardinal and $\operatorname{supp}(\dot{f}_\xi)$ is a countable subset of κ, for every $\xi < \kappa$, the family all $\operatorname{supp}(\dot{f}_\xi)$ necessarily has cardinality κ; otherwise, $\bigcup_{\xi<\kappa} \operatorname{supp}(\dot{f}_\xi)$ would be a bounded set in κ and using that κ is strongly inaccessible there are not enough countable subsets of a bounded subset of κ, therefore there are κ-many ξ's for which the supports of $\dot{f}_\xi$ coincide. However, $\mathbb{P}\restriction A$ cannot code κ distinct functions for any countable $A\subseteq\kappa$. Henceforth, we assume that $\operatorname{supp}(\dot{f}_\xi)$ are all distinct.

As κ is a regular cardinal, we can also apply the Δ-system lemma to the family of supports of those good $\mathbb{P}$-names $\dot{f}_\xi$'s. Thus we assume that $\{\operatorname{supp}(\dot{f}_\xi)\colon \xi<\kappa\}$ forms Δ-system with root R. Moreover, due to R is at most countable, there must be κ-many ξ's such that the $\mathbb{P}$-names $\dot{f}_\xi$ are exactly the same name over R. Since there are only $\aleph_2$-many possible types for those $\dot{f}_\xi$ good $\mathbb{P}$-names we can assume further that any two $\dot{f}_\xi$ and $\dot{f}_\eta$ are isomorphic names for $\xi,\eta<\kappa$.

Let M be a countable elementary submodel of $H(\vartheta)$ for some large enough regular cardinal ϑ such that $\mathbb{P}$ and $\{\dot{f}_\xi\colon \xi<\kappa\}$ are elements of M. Let $\dot{W}$ be a $\mathbb{P}$-name for a G_δ-subset of ω^ω such that $\mathbb{P}\Vdash X\cap\dot{W}\supseteq\{\dot{f}_\xi\colon \xi\in M\cap\kappa\}$. Fix $\mathbb{P}$-names $\dot{W}_m$ for open subsets of ω^ω, $m\in\omega$, such that $\mathbb{P}\Vdash\dot{W}=\bigcap_{m\in\omega}\dot{W}_m$. Consider now another countable elementary submodel N of $H(\vartheta)$ containing everything that is relevant for the proof and such that $M\in N$ and the $\mathbb{P}$-names $\dot{W}$, $\dot{W}_n$ are all elements of N. Inside N, we can fix an enumeration $\{\xi_n\colon n\in\omega\}$ of $M\cap\kappa$.

Working inside N, for every $m\in\omega$, we can find a $\mathbb{P}$-name $\dot{g}_m\in N$ for a function from ω into ω such that

$$\mathbb{P}\Vdash\big[\dot{f}_{\xi_n}\restriction\dot{g}_m(n)\big]\subseteq\dot{W}_m,$$

for every $n\in\omega$.

Fix $\mathbb{P}$-names $\dot{U},\dot{U}_m\in N$, for $m\in\omega$, such that $\mathbb{P}\Vdash\dot{U}_m=\bigcup_{n\in\omega}[\dot{f}_{\xi_n}\restriction\dot{g}_m(n)]$ and $\mathbb{P}\Vdash\dot{U}=\bigcap_{m\in\omega}\dot{U}_m$. Then U is forced to be a G_δ subset of W. To finish the proof we prove the following claim:

Claim 1. *If $\xi\in\kappa\setminus N$ and $\operatorname{supp}(\dot{f}_\xi)\cap N=\emptyset$, then $\mathbb{P}\Vdash\dot{f}_\xi\in\dot{U}$.*

In order to establish the claim let ξ be any such ordinal outside N and fix $m\in\omega$. It suffices to show that $\dot{f}_\xi$ is forced by a dense set of conditions to be in $\dot{U}_m$. So, fix a condition $r\in\mathbb{P}$. Without loss of generality, assume r is a good $\mathbb{P}$-condition.

Firstly, recall that $\dot{f}_\xi$ is a good $\mathbb{P}$-name for a function in ω^ω, say

$$\dot{f}_\xi=\bigcup_{i\in\omega}\big\{\big\langle p,\langle i,j_i^p\rangle\big\rangle\colon\ p\in\mathcal{A}_i\ \&\ j_i^p\in\omega\big\}.$$

We are assuming that $\{\operatorname{supp}(\dot{f}_\xi)\colon \xi<\kappa\}$ forms a Δ-system, by elementarity we can assume that its root $R\in M$, hence $R\subseteq M$. As $\operatorname{supp}(r)$ is finite there is an $n\in\omega$ such that

$$\operatorname{supp}(r)\cap\big(\operatorname{supp}(\dot{f}_{\xi_n})\setminus R\big)=\emptyset.$$

Suppose that the good $\mathbb{P}$-name $\dot{f}_{\xi_n}$ is $\bigcup_{i\in\omega}\{\langle q,\langle i,j_i^q\rangle\rangle : q\in\mathcal{A}_i'\ \&\ j_i^q\in\omega\}$. Since we are assuming that $\dot{f}_\xi$ and $\dot{f}_{\xi_n}$ are isomorphic names, there is a bijection $\Phi:\operatorname{tr}\operatorname{supp}(\dot{f}_\xi)\to\operatorname{tr}\operatorname{supp}(\dot{f}_{\xi_n})$ witnessing that $\dot{f}_\xi\approx\dot{f}_{\xi_n}$. By definition, Φ induces bijective functions ψ_α from the column α of $\operatorname{tr}\operatorname{supp}(\dot{f}_\xi)$ onto the column $\varphi(\alpha)$ of $\operatorname{tr}\operatorname{supp}(\dot{f}_{\xi_n})$. For $\alpha\in\operatorname{supp}(\dot{f}_\xi)\cap\operatorname{supp}(r)$, the function ψ_α can be extended to include $\operatorname{tr}\operatorname{supp}(r(\alpha))$ in its domain preserving the ordering between ordinals.[4] Denote that extension of ψ_α again by ψ_α. Define a new condition $r_0\in\mathbb{P}$ by

$$r_0(\alpha')=\begin{cases} r(\alpha') & \text{if } \alpha'\in N\setminus(\operatorname{supp}(\dot{f}_{\xi_n})\setminus R),\\ \overline{\psi_\alpha}(r) & \text{if } \alpha'=\varphi(\alpha)\ \&\ \alpha\in\operatorname{supp}(\dot{f}_\xi)\cap\operatorname{supp}(r),\\ 1_{\mathbb{P}_{\mu_{\alpha'}}} & \text{otherwise,}\end{cases}$$

for all $\alpha'<\kappa$. Then $r_0\leqslant r\restriction N$. Now extend r_0 to some condition r_1 such that $\operatorname{supp}(r_1)\subseteq N$ and

(1) r_1 decides the value of $\dot{g}_m(n)$; that is, $r_1\Vdash\dot{g}_m(n)=k$,
(2) r_1 decides $\dot{f}_{\xi_n}\restriction k$; that is, $r_1\Vdash f_{\xi_n}\restriction k=t$ for some $t:k\to\omega$,
(3) r_1 is a good condition.

[4] It is here that clauses (2a) and (2b) of Definition 3.3 play their role in the definition of isomorphic names. Note that the extension of ψ_α may not be an isomorphism of simple templates though.

Once again, for each $\varphi(\alpha) \in \operatorname{supp}(\dot{f}_{\xi_n}) \cap \operatorname{supp}(r_1)$, the function ψ_α can be extended in such a way that $\operatorname{tr}\operatorname{supp}(r_1(\varphi(\alpha)))$ is included in its range and the ordering between ordinals is still preserved. One more time, we denote that extension again by ψ_α. Define another new condition $r_2 \in \mathbb{P}$ by

$$r_2(\alpha) = \begin{cases} r_1(\alpha) & \text{if } \alpha \in N, \\ \overline{\psi_\alpha^{-1}}(r_1) & \text{if } \varphi(\alpha) \in \operatorname{supp}(\dot{f}_{\xi_n}) \cap \operatorname{supp}(r_1), \\ r(\alpha) & \text{otherwise,} \end{cases}$$

for all $\alpha < \kappa$. Then the condition r_2 has the following properties:

(1) $r_2 \leqslant r_1$ and $r_2 \leqslant r$,
(2) $r_2 \Vdash \dot{f}_\xi \restriction k = \dot{f}_{\xi_\alpha} \restriction k$, and
(3) $r_2 \Vdash \dot{f}_\xi \in U_m$.

This completes the proof of Lemma 2.7 and hence it completes the proof of Theorem 2.3 as well. □

At first glance, the fact that κ is a strongly inaccessible cardinal does not seem to be important and one could think that the method might work as well for a singular cardinal κ. Of course, we know that it cannot be true because of Theorem 2.1. We can even give a simple $\mathbb{P}$-name for a λ-set of maximal cardinality. Suppose that $\mu = \operatorname{cf}(\kappa)$ and consider a strictly increasing sequence $\langle \kappa_\alpha \colon \alpha < \mu \rangle$ cofinal in κ and such that $\kappa_\alpha > \mu$ for all $\alpha < \mu$. Let $\dot{f}_{\alpha,\beta}$ be a $\mathbb{P}$-name for the βth generic real added by $\mathbb{P}_\alpha$ $(\beta < \alpha)$ and let $\dot{h}_{\alpha,\beta,\gamma,\delta}$ be such that $\mathbb{P} \Vdash \dot{h}_{\alpha,\beta,\gamma,\delta} = \max\{\dot{f}_{\alpha,\gamma}, \dot{f}_{\beta,\delta}\}$, for all $\alpha, \gamma < \mu$, $\kappa_\alpha < \beta < \kappa_{\alpha+1}$ and $\delta < \kappa_{\alpha+1}$. Then $\operatorname{supp}(\dot{h}_{\alpha,\beta,\gamma,\delta}) = \{\gamma, \beta\}$ and by genericity we are considering $\mathfrak{c} = (\kappa)^{\mathbf{V}}$ distinct functions. However, there is no Δ-system of κ-many distinct supports.

Acknowledgements

I am very grateful to Michael Hrušák for suggesting the research here contained, sharing his insight and clever ideas not only about what is written here but always.

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