WHEN IS $\mathbb{R}$ THE UNION OF AN INCREASING FAMILY OF NULL SETS?

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Abstract. We study the problem in the title and show that it is equivalent to the fact that every set of reals is an increasing union of measurable sets. We also show the relationship of it with Sierpiński sets.

1. Introduction

The study of this note was motivated by the determination of the existence of solutions in a Markov decision problem with constraints (see [Piu97] for this topic). The problem we faced was to find an optimal stochastic kernel supported on a measure function. This led us to try to extend the domain of a measurable function on the union of a well ordered family of measurable sets. However the union of such a family may not be measurable: If $(X, \mathcal{A})$ is a measure space such that every one-point set is measurable but not all subsets of X are, then there is a family of measurable sets well ordered by containment and whose union is not measurable. Indeed, picking $A \subseteq X$ which is not measurable and using the Axiom of Choice one can well order the set A and then take the first initial segment of A which is also not measurable. Then taking the initial segments of A will give us the natural well ordered family of measurable sets whose union is not measurable.

Our curiosity took us to ask ourselves when are there subsets which cannot be written as an increasing union of measurable sets? It is not hard to see that the question in a general framework is trivial since one can consider the σ-algebra generated by countable subsets of ω_2 —the second uncountable ordinal, then the set of limit ordinals in ω_2 cannot be written as increasing union of measurable sets.

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Our research grew trying to understand the answer of the following question:

Question 1. Consider the measurable space $(\mathbb{R}, \mathcal{M})$, where $\mathcal{M}$ is the Lebesgue σ-algebra, let A be a subset of $\mathbb{R}$. Is it always possible to express A as union of a family $\{A_\alpha : \alpha \in \kappa\} \subseteq \mathcal{M}$ such that $\alpha < \beta$ implies $A_\alpha \subseteq A_\beta$?

Below it is shown that the answer cannot be decided in ZFC, the usual axiomatic system for set theory. For example it follows that both CH and $\mathsf{MA} + \neg\mathsf{CH}$ imply that the answer is positive. Nevertheless, a Sierpiński set of cardinality $\aleph_2$ would be a counterexample. The existence of a set A which cannot be written as an increasing union of measurable sets seem very closed to the existence of a Sierpiński set. M. Hrušák then asked us

Question 2. Does the existence of a subset of $\mathbb{R}$ which cannot be written as an increasing union of measurable sets imply the existence of a Sierpiński set?

In the last section we show that is not the case in the generic extension obtained by the iteration of random and Miller forcings.

Although it is possible to write sketchy proofs for most of our propositions, we would like to address this note to a wider set of readers giving more detailed proofs. Our notation is standard. For example, Greek letters $\alpha, \beta, \gamma, \xi, \eta$, etc. represent ordinal numbers. The elements of an ordinal number α are the ordinal numbers β which are strictly less than α. Cardinal numbers are the initial ordinal numbers and Greek letters κ, θ, ν, etc. represent cardinal numbers. The symbol $\mathfrak{c}$ represents the cardinality of the set of real numbers and it is well known that $\mathfrak{c} = 2^{\aleph_0}$. We reserve the symbol $\mathbf{V}$ to denote the ground model in forcing arguments. The set of rational numbers will be denoted by $\mathbb{Q}$ and the tree $\omega^{<\omega}$ is the set of all $s : n \to \omega$, for $n \in \omega$, ordered by extension and $|s|$ will be the length of the sequence $s \in \omega^{<\omega}$. All undefined set theoretical notions can be found in [BJ95] or in [Kun83].

There are several equivalent formulations of Martin's Axiom MA; one of the simplest is: *Suppose X is a Hausdorff compact topological space in which every family of pairwise disjoint non-empty open sets is at most countable. If $\mathcal{U}$ is a non-empty family of dense open sets in X and $|\mathcal{U}| < \mathfrak{c}$, then $\bigcap \mathcal{U} \neq \emptyset$.*

Expressed in this way MA is a generalization of the well-known Baire's Theorem. If CH holds then MA is just Baire's Theorem; thus when one tries to get more out of Martin's Axiom one also assume that CH does not hold and hence we commonly assume $\mathsf{MA} + \neg\mathsf{CH}$ instead of only MA.

With respect to measure theory, measurable set from here on will refer to subsets of $\mathbb{R}$ which are measurable with respect to the Lebesgue measure on $\mathbb{R}$, which we will denote by λ. We will also need to use the outer Lebesgue measure and the Lebesgue inner measure which are defined by

$$\lambda^*(A) = \inf\{\lambda(U) : A \subseteq U \ \&\ U \text{ is open}\}, \text{ and}$$
$$\lambda_*(A) = \sup\{\lambda(K) : K \subseteq A \ \&\ K \text{ is compact}\}$$

for all $A \subseteq \mathbb{R}$.

Many of our arguments will involve *null sets*; that is, measure zero subsets of $\mathbb{R}$. The family $\mathcal{N}$ of null sets form an σ-ideal of subsets of $\mathbb{R}$. This ideal has some classic cardinal coefficients that we define now:

$$\begin{aligned}
\mathsf{add}\,(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \;\&\; \bigcup \mathcal{A} \notin \mathcal{N}\},\\
\mathsf{cov}\,(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \;\&\; \bigcup \mathcal{A} = \mathbb{R}\},\\
\mathsf{non}\,(\mathcal{N}) &= \min\{|Y| : Y \subseteq \mathbb{R} \;\&\; Y \notin \mathcal{N}\}, \text{ and}\\
\mathsf{cof}\,(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \;\&\; (\forall N \in \mathcal{N})\,(\exists A \in \mathcal{A})\,(N \subseteq A)\}.
\end{aligned}$$

The relationship between these cardinals can be expressed by a simple diagram where the arrows denote $\leq$. There is a huge amount of research on the relationship of this cardinal among themselves and among other cardinal invariants of the continuum. See for example [BJ95].

$$\begin{tikzcd}
 & & \mathsf{non}\,(\mathcal{N}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \mathsf{add}\,(\mathcal{N}) \arrow[ur] \arrow[dr] & & \mathsf{cof}\,(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathsf{cov}\,(\mathcal{N}) \arrow[ur] & &
\end{tikzcd}$$

The diagram represent all possible inequalities provable in ZFC. Only the first inequality is probable not to be an equality, all other are consistently equalities. For example the continuum hypothesis CH imply that the four cardinals above coincide and are equal to $\aleph_1$. Martin's Axiom plus the negation of CH also implies that those cardinal are equal and they are equal to $\mathfrak{c}$ which under this hypothesis is different to $\aleph_1$.

2. $\mathcal{N}$-INACCESSIBLE SETS

We start by showing that to study the behaviour of the increasing families of measurable sets one can restric to those families consisting of null sets; this was observed by M. Hrušák [Hru07].

Theorem 2.1. *The following are equivalent:*

(1) *Every set $X \subseteq \mathbb{R}$ can be written as an increasing union of measurable sets.*
(2) *Every set $X \subseteq \mathbb{R}$ can be written as an increasing union of null sets.*
(3) *$\mathbb{R}$ can be written as an increasing union of null sets.*

Proof. Clearly (2) and (3) are equivalents and (3) implies (1). We are left to prove (1) implies (3). In order to do that recall that a Vitali set is a classical example of a non-measurable set (see [Roy88]). Let X be a Vitali set. This set has two interesting properties for us: $\lambda_*\,(X) = 0$ and $X + \mathbb{Q} = \mathbb{R}$. On the other hand, by hypothesis there is a family $\{A_\alpha : \alpha < \kappa\}$ of measurable sets such that $\alpha < \beta$ implies $A_\alpha \subseteq A_\beta$ for all $\alpha, \beta < \kappa$ and such that $X = \bigcup_{\alpha<\kappa} A_\alpha$. Then A_α is a null set for all $\alpha < \kappa$ and so is $N_\alpha = A_\alpha + \mathbb{Q}$. Moreover, $\mathbb{R} = \bigcup_{\alpha<\kappa} N_\alpha$ as needed. ❏

In order to save some words from now on we introduce the following definition:

Definition 2.2. A subset $X \subseteq \mathbb{R}$ is call $\mathcal{N}$-*inaccessible* if $X \neq \bigcup_{\alpha<\kappa} N_\alpha$ whenever $\{N_\alpha : \alpha < \kappa\}$ is an increasing family of null sets contained in X.

Now, we show that some relations between the cardinal coefficients of the ideal $\mathcal{N}$ imply that no subset of $\mathbb{R}$ is $\mathcal{N}$-inaccessible.

Proposition 2.3. *If* $\mathsf{cov}(\mathcal{N}) = \mathsf{add}(\mathcal{N})$ *or* $\mathsf{non}(\mathcal{N}) = \mathfrak{c}$*, then no* $X \subseteq \mathbb{R}$ *is* $\mathcal{N}$*-inaccessible.*

Proof. By Theorem 2.1 all we need to show is that $\mathbb{R}$ is not $\mathcal{N}$-inaccessible. Let $\kappa = \mathsf{cov}(\mathcal{N})$. There is a family $\{N_\alpha : \alpha < \kappa\} \subseteq \mathcal{N}$ such that $\mathbb{R}$ is the union of such family. Since this family is not necessarily well ordered by containment we can define a new family letting

$$N'_\alpha = \bigcup_{\beta \leq \alpha} N_\beta,$$

for every $\alpha < \kappa$. If $\mathsf{cov}(\mathcal{N}) = \mathsf{add}(\mathcal{N})$, it follows that $N'_\alpha \in \mathcal{N}$ for all $\alpha < \kappa$. Thus $\mathbb{R} = \bigcup_{\alpha<\kappa} N'_\alpha$ and $\alpha < \beta < \kappa$ implies $N'_\alpha \subseteq N'_\beta$ as we wanted. Similarly, if $\mathsf{cov}(\mathcal{N}) = \aleph_1$, it is easy to get an increasing family of null sets whose union is $\mathbb{R}$.

If $\mathsf{non}(\mathcal{N}) = \mathfrak{c}$ and we enumerate $\mathbb{R}$ as $\{x_\alpha : \alpha < \mathfrak{c}\}$, then $M_\alpha = \{x_\beta : \beta \leq \alpha\}$ is a set of cardinality strictly less than $\mathfrak{c}$ and hence a null set. Clearly the family $\{M_\alpha : \alpha < \kappa\}$ witness that $\mathbb{R}$ is not $\mathcal{N}$-inaccessible. ❑

Corollary 2.4. $\mathsf{MA} + \neg\mathsf{CH}$ *or* CH *imply there are no* $\mathcal{N}$*-inaccessible subsets of* $\mathbb{R}$.

The interesting thing about Proposition 2.3 is that it gives us every subset of $\mathbb{R}$ as an increasing union of null sets if one of the hypothesis is met, no matter how big the continuum is. There are models for ZFC for which $\mathsf{cov}(\mathcal{N}) = \mathsf{add}(\mathcal{N}) < \mathfrak{c}$. (See [BJ95].)

On the other hand, if $\mathsf{non}(\mathcal{N})$ is smaller than the cofinality of $\mathsf{cov}(\mathcal{N})$ then every set $X \subseteq \mathbb{R}$, with $\lambda(X) > 0$, is an $\mathcal{N}$-inaccessible. Indeed, if $\{N_\alpha : \alpha < \kappa\}$ is an increasing family of null subsets of X, then $\kappa \geq \mathsf{cov}(\mathcal{N})$ and there is $Y \subseteq X$ which is not null and $|Y| = \mathsf{non}(\mathcal{N})$. But such a set Y must be contained in some N_α, which is a contradiction.

3. Sierpiński sets and $\mathcal{N}$-inaccessible sets

For the remainder of the paper we shall use forcing arguments. The method of forcing used to prove the relative consistency of some statements was introduced by P. Cohen in the 60's and since then it has become the most important tool in set theory. We suggest to see [Kun83] to learn the basic facts about forcing.

Before to introduce the forcing that gives us a $\mathcal{N}$-inaccessible subset of $\mathbb{R}$ we will introduce a special type of subsets of the reals.

Definition 3.1. An uncountable set X of real numbers is a *Sierpiński set* if $X \cap N$ is at most countable for every null set $N \subseteq \mathbb{R}$.

Observe that a Sierpiński set is not measurable because if it were measurable, then $\lambda(X) > 0$ and in such case X must contain a closed subset F of positive measure. However, every G_δ subset of $\mathbb{R}$ of positive measure contains a homeomorphic copy of the Cantor set and of measure zero (see [Oxt80, Lemma 5.1]). But this fact contradicts that X is a Sierpiński set.

Sierpiński sets do not necessarily exist. For example, if $\mathsf{MA} + \neg\mathsf{CH}$ holds, there are no Sierpiński sets. However, CH implies that Sierpiński sets do exist. To see this last part, using CH, find a family of null sets $\{N_\xi : \xi < \omega_1\}$ such that

- $N_\eta \subseteq N_\xi$ for $\eta \leq \xi < \omega_1$ and
- for every null set N there is $\xi < \omega_1$ such that $N \subseteq N_\xi$.

Choosing $x_\xi \in \mathbb{R} \setminus \bigcup_{\eta<\xi} N_\eta$ for $\xi < \omega_1$, the set $\{x_\xi : \xi < \omega_1\}$ will be a Sierpiński set.

Now we define the natural forcing to add a Sierpiński set of any given cardinality.

Definition 3.2. *Random forcing* $\mathbb{B}$ is defined as the set

$$\{B \subseteq [0,1] : B \text{ is a Borel set and } \lambda(B) > 0\}$$

with the ordering $B_0 \leq B_1$ in case that $B_0 \subseteq B_1$.

Random forcing $\mathbb{B}$ satisfies the countable chain condition (c.c.c.), hence it does not collapse cardinals. For this and other properties of $\mathbb{B}$ see [Kun83] and [BJ95]. Here we only say that $\mathbb{B}$ adds a real number r, called the *random real* over $\mathbf{V}$, that is not a member of any null set which can be coded in the ground model $\mathbf{V}$. Denote by $\mathbb{B}(\kappa)$ the forcing which adds κ-many random reals to a model $\mathbf{V}$. The important properties of $\mathbb{B}(\kappa)$ for us is concentrated in the following lemma stated and proven in [BJ95, p. 438].

Lemma 3.3. *Let $G = \{r_\xi : \xi < \kappa\}$ be the generic sequence of κ random reals added by $\mathbb{B}(\kappa)$ to a model $\mathbf{V}$. Then $\{r_\xi : \xi < \theta\}$ is a Sierpiński set in the model $\mathbf{V}[G]$, for every $\aleph_1 \leq \theta \leq \kappa$.*

This lemma is all we need to get an $\mathcal{N}$-inaccessible set.

Theorem 3.4. *In the model obtained by adding $\aleph_2$ random reals to a model $\mathbf{V}$ of a rich enough segment of ZFC, there is an $\mathcal{N}$-inaccessible set.*

Proof. Let $X = \{r_\xi : \xi < \omega_2\}$ be the set of random reals over $\mathbf{V}$ and let $\{A_\alpha : \alpha < \kappa\}$ be an increasing family of null subsets of X such that $X = \bigcup_{\alpha<\kappa} A_\alpha$. Since $|X| = \aleph_2$, it follows that there is $\alpha < \kappa$ such that $|A_\alpha| = \aleph_1$. Since a subset of a Sierpiński set is a Sierpiński set, from Lemma 3.3 it follows that A_α must be a Sierpiński set and hence non-measurable. Thus X is not representable as union of increasing family of null sets. ❑

4. $\mathcal{N}$-INACCESSIBLE SETS WITHOUT SIERPIŃSKI SETS

In this section we shall answer Hrušák's question: Does the existence of a subset of $\mathbb{R}$ which cannot be written as an increasing union of measurable sets implies the existence of a Sierpiński set?

We shall use a countable support iteration of proper forcings, one of them will be the random forcing that was introduced in the previous section and the *rational perfect set forcing* that we introduce now. For convenience we are going to change the set of all real numbers for the unit interval $[0,1]$.

Definition 4.1. The *Miller forcing* (or rational perfect set forcing) is the following forcing notion: $T \in \mathbb{PT}$ if and only if

(1) $T \subseteq \omega^{<\omega}$,
(2) T is a tree,
(3) $(\forall s \in T)\,(s \text{ is increasing})$,
(4) $(\forall s \in T)\,(\exists t \in T)\,(\exists^\infty n \in \omega)\,(s \subseteq t^\frown \langle n \rangle \in T)$,

with the ordering $\leq$ defined by $T \leq S$ if and only if $T \subseteq S$.

If $G \subseteq \mathbb{PT}$ is generic filter over a model $\mathbf{V}$, then $m = \bigcap G \in \omega^\omega$ is called a Miller real. $\mathbb{PT}$ is a proper forcing which preserves outer Lebesgue measure. The following theorem was proven in [JS94]:

Theorem 4.2 (Judah, Shelah)**.** *If $Y \in \mathbf{V}$ is a Sierpiński set, then Y is not a Sierpiński set in any extension of $\mathbf{V}$ containing a Miller real over $\mathbf{V}$.*

Consider the following countable support iteration $\mathbb{P} = \left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \right\rangle$ over a ground model $\mathbf{V}$ where CH holds: for each $\alpha < \omega_2$,

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \simeq \mathbb{B}, \text{ when } \alpha \text{ is even}$$
$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \simeq \mathbb{PT}, \text{ when } \alpha \text{ is odd.}$$

Since $\mathbb{B}$ and $\mathbb{PT}$ are proper forcing, $\mathbb{P}$ is a proper forcing as well. If $G \subseteq \mathbb{P}$ is a generic filter over $\mathbf{V}$ then letting $G_\alpha = G \cap \mathbb{P}_\alpha$ we obtain a generic filter over $\mathbf{V}$ and $\mathbf{V}[G_\alpha]$ is a submodel of $\mathbf{V}[G]$.

Theorem 4.3. *If $G \subseteq \mathbb{P}$ is a generic filter over $\mathbf{V}$, then in the model $\mathbf{V}[G]$ there are no Sierpiński sets and there are $\mathcal{N}$-inaccessible sets.*

Proof. To show that there are no Sierpiński sets in $\mathbf{V}[G]$ proceed by contradiction and assume that $X \in \mathbf{V}[G]$ is a Sierpiński set. Without loss of generality, assume that $|X| = \aleph_1$. Since every real number in $\mathbf{V}[G]$ depends only on countable many steps of the forcing iteration, there is a odd ordinal $\alpha < \omega_2$ such that $X \in \mathbf{V}[G_\alpha]$. Then Theorem 4.2 gives us that X is not a Sierpiński set in $\mathbf{V}[G_{\alpha+1}]$, and hence it is not in the final extension $\mathbf{V}[G]$.

Let $X = \{r_\alpha : \alpha < \omega_2 \ \&\ \alpha \text{ is even}\}$; that is, X is the set of all random reals over $\mathbf{V}$ added by $\mathbb{P}$. From Theorems 6.3.12, 7.3.47 and 6.3.13 in [BJ95], it follows that $\mathbb{P}$ preserves outer Lebesgue measure. Since $\lambda^*(\mathbf{V} \cap [0,1]) = 1$ then $\lambda^*([0,1] \setminus X) = 1$, and hence $\lambda_*(X) = 0$.

Now we can get two helpful properties of X. First, no subset of X of cardinality $\aleph_2$ is measurable. Indeed, if $Y \subseteq X$ has cardinality $\aleph_2$ and it were measurable, then there would be a Borel null set $N \in \mathbf{V}[G]$ such that $Y \subseteq N$. Since Borel sets are coded by real numbers, there must be $\alpha < \omega_2$ such that $N \in \mathbf{V}[G_\alpha]$. However, any random real $r_\beta \in Y$ for $\beta > \alpha$ will not be an element of N, contradicting that N contains Y. Second, every $A \subseteq X$ of cardinality $\aleph_1$ contains a non-measurable set. To see this, fix $A \subseteq X$ with $|A| = \aleph_1$ and let

$$\alpha = \min\{\beta < \omega_2 : |A \cap \{r_\gamma : \gamma < \beta\}| = \aleph_1\}.$$

Clearly α has cofinality ω_1. Put $B = \{r_\gamma : \gamma < \alpha\}$. If B is measurable, then there is a Borel null set N such that $B \subseteq N$. Since real numbers are not added at stages of uncountable cofinality in proper forcing iterations, there must be $\beta < \alpha$ such that $N \in \mathbf{V}[G_\beta]$. Again taking γ such that $\beta < \gamma < \alpha$ we found $r_\gamma \in B \setminus N$, a contradiction.

Now suppose X can be written as a union (not necessarily increasing) of measurable sets, then either $X = \bigcup_{\alpha<\omega_1} A_\alpha$ or $X = \bigcup_{\alpha<\omega_2} A_\alpha$. Nevertheless, neither of the options is possible: in the first one, there must be an $\alpha < \omega_1$ such that $|A_\alpha| = \aleph_2$, but we know that A_α cannot be measurable. The second option is not possible either, because each uncountable A_α must be of measure zero but it must contain a non-measurable set, which is a contradiction. ❑

In the Miller model; that is, the generic model obtained by an ω_2-iteration with countable support of Miller forcing there are no Sierpiński sets but in that model there are no $\mathcal{N}$-inaccessible sets due to the equality $\mathsf{add}(\mathcal{N}) = \mathsf{cov}(\mathcal{N})$ in that model.

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