Characteristic subgroups are not preserved by isomorphisms of tables of marks

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We construct two non-isomorphic groups G and Q of order 96 which have isomorphic tables of marks, but such that the centre of G is mapped to a non-characteristic subgroup of Q.

Let G, Q be finite groups. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. The matrix whose H, K-entry is $\#(K^H)$ is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$).

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G. It is easy to see that if G and Q have isomorphic tables of marks, then they have isomorphic Burnside rings; the converse is an open problem.

Definition 1. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#(K'^{H'}) = \#(K^H)$ for all subgroups H, K of G.

Two non-isomorphic groups of order 96 with isomorphic tables of marks

Let S_3 be the symmetric group or order 6. Let C_8 be the cyclic group of order 8, generated by x, and let C_2 be the cyclic group of order 2, generated by y.

Let δ be the only non-trivial homomorphism from S_3 to C_8. Let W denote the group $S_3 \times C_8$. Let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, and let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$.

Since α has order two, we can define the group G as the semidirect product of W with C_2 by α, that is, in G we have that $y(\lambda, x^i)y = \alpha(\lambda, x^i)$. Similarly, we define the group Q as the semidirect product of W and C_2 by β; in Q we have that $y(\lambda, x^i)y = \beta(\lambda, x^i)$. We shall denote the elements of both G and Q as $\lambda x^i y^j$.

Note that in G, x and y commute, and the centre of G is therefore the subgroup generated by x, which is a subgroup of order 8; however, x and y do not commute in Q, and the centre of Q is the subgroup generated by x^2, which is a subgroup of order 4. In particular, we also have that G and Q are non-isomorphic groups of order 96.

Theorem 2. *Let S be a subset of G (and therefore S is also a subset of Q). Then S is a subgroup of G if and only if S is a subgroup of Q. Moreover, two subgroups are conjugate in G if and only if they are conjugate in Q, and the identity map on the family of conjugacy classes of subgroups defines and isomorphism between the tables of marks of G and Q.*

Proof: See *Two non-isomorphic groups of order 96 with isomorphic tables of marks and non-corresponding centres and abelian subgroups*, Communications in Algebra, 2009.

Theorem 3. *The subgroup of Q generated by x is not a characteristic subgroup. In particular, the isomorphism of tables of marks between G and Q maps the centre of G to a non-characteristic subgroup of Q.*

Proof: We construct an automorphism of Q that does not preserve the subgroup generated by x. Let $\eta : Q \longrightarrow Q$ be given by

$$\eta(\lambda x^i y^j) = \lambda x^{3i+6i^2+\left(1-Sgn(\lambda)\right)(2i+3)} y^{i+j+\frac{1-Sgn(\lambda)}{2}}$$

We claim that for a generator g of Q and an arbitrary $\lambda x^i y^j$ we have that $\eta(g\lambda x^i y^j) = \eta(g)\eta(\lambda x^i y^j)$, where g can be $(1,2),(1,2,3),x,y$, so η is indeed a homomorphism.

$g = (1,2)$:

$$\eta\Big((1,2)(\lambda x^i y^j)\Big) = \eta\Big((1,2)\lambda x^i y^j\Big)$$
$$= (1,2)\lambda x^{5i+6i^2-2iSgn\big((1,2)\lambda\big)-3sgn\big((1,2)\lambda\big)+3}$$
$$y^{i+j+\frac{1-Sgn\big((1,2)\lambda\big)}{2}}$$
$$= (1,2)\lambda x^{5i+6i^2+2iSgn(\lambda)+3sgn(\lambda)+3} y^{i+j+\frac{1+Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta\Big((1,2)\Big)\eta\Big(\lambda x^i y^j\Big)$$
$$= \Big((1,2)x^6 y\Big)\Big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\Big)$$
$$= (1,2)\lambda$$
$$x^{6+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))}$$
$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$
$$= (1,2)\lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+7} y^{1+i+j\frac{1-Sgn(\lambda)}{2}}$$

These two expressions coincide, because:

$$\Big(5i + 2iSgn(\lambda) + 3sgn(\lambda) + 3\Big) -$$
$$\Big(i - 2iSgn(\lambda) - Sgn(\lambda) + 7\Big)$$
$$= 4i + 4isgn(\lambda) + 4sgn(\lambda) + 4$$
$$= 4\Big((1 + Sgn(\lambda))(i + 1)\Big)$$

$g=(1,2,3)$:

$$\eta\Big((1,2,3)(\lambda x^i y^j)\Big)=\eta\Big((1,2,3)\lambda x^i y^j\Big)$$

$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn\Big((1,2,3)\lambda\Big)-3Sgn\Big((1,2,3)\lambda\Big)+3}$$

$$y^{i+j+\frac{1-Sgn\Big((1,2,3)\lambda\Big)}{2}}$$

$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta\Big((1,2,3)\Big)\eta\Big(\lambda x^i y^j\Big)$$

$$=\Big((1,2,3)\Big)\Big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$

$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\Big)$$

$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$

$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$g = y$:

$$\eta(y\lambda x^i y^j) = \eta(\lambda x^{5i+2-2Sgn(\lambda)} y^{1+j})$$

$$= \lambda x^{5[5i+2-2Sgn(\lambda)]+6[5i+2-2Sgn(\lambda)]^2-}$$

$$2[5i+2-2Sgn(\lambda)]Sgn(\lambda)-3Sgn(\lambda)+3 \, y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$$= \lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta(y)\eta(\lambda x^i y^j) = y\Big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$

$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\Big)$$

$$= \lambda x^{5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2[1-Sgn(\lambda)]}$$

$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$$= \lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$g = x$

$$\eta(x\lambda x^i y^j) = \eta(\lambda x^{1+i} y^j)$$
$$= \lambda x^{5(1+i)+6(1+i)^2-2(1+i)Sgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$
$$= \lambda x^{i+6i^2-2iSgn(\lambda)-5Sgn(\lambda)+6} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta(x)\eta(\lambda x^i y^j) = (xy)\Big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\Big)$$
$$= \lambda x^{1+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))}$$
$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$
$$= \lambda x^{2+i+6i^2-2iSgn(\lambda)-Sgn(\lambda)} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

These two expressions coincide because:

$$\Big(6 - 5Sgn(\lambda)\Big) - \Big(2 - Sgn(\lambda)\Big) = 4(1 - Sgn(\lambda))$$

Therefore η is a group homomorphism.

Moreover,

$$(1,2) = \eta((1,2)x^6y), \qquad (1,2,3) = \eta(1,2,3),$$

$$x = \eta(xy), \qquad y = \eta(y)$$

so η must be an automorphism. Finally, note that $\eta(x) = xy$, so the subgroup generated by x is not a characteristic subgroup of Q.