

ON SOME INVARIANTS PRESERVED BY ISOMORPHISMS OF TABLES OF MARKS

Luis M. Huerta-Aparicio, Ariel Molina-Rueda, Alberto G. Raggi-Cárdenas
(*), Luis Valero-Elizondo
Facultad de Ciencias Físico-Matemáticas, UMSNH,
(*), Instituto de Matemáticas, UNAM

Abstract

Let G and Q be groups with isomorphic tables of marks, and for each subgroup H of G , let H' denote a subgroup of Q assigned to H under an isomorphism between the tables of marks of G and Q . We prove that if H is cyclic/elementary abelian/maximal/the Frattini subgroup/the commutator subgroup, then H' has the same property. However, we give examples where H is abelian and H' is not, and where H is the centre of G and H' is not the centre of Q .

Short title: Tables of marks. ¹

1 Introduction.

Groups with isomorphic tables of marks may not be isomorphic groups (as proved by Thévenaz in [5]), but one still expects them to have many attributes in common. Indeed, if G and Q are groups with isomorphic tables of marks, then they have isomorphic composition factors (see [3]), and they also have isomorphic Burnside rings (the converse is still an open problem, put forward also in [3]); if two groups have isomorphic Burnside rings and one of them is abelian/Hamiltonian/minimal simple, then the two groups are isomorphic (see [4]), and a similar result is known for several families of simple groups (see [2]). The key for the previous proofs is that a lot can be said about a group in terms of its table of marks (or even in terms of its

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Burnside ring). But how much can be said about a group's **subgroups** by its table of marks? Is it possible to tell when a subgroup is cyclic, elementary abelian or abelian? Can we tell from the table of marks of G which subgroup is the centre of G , or its Frattini subgroup, or its commutator subgroup? More generally, if G and Q are groups with isomorphic tables of marks, this isomorphism establishes a correspondence between the (conjugacy classes of) subgroups of G and Q . Does this correspondence preserve cyclic/elementary abelian/abelian subgroups? Does the centre/Frattini subgroup/commutator subgroup of G correspond with its counterpart in Q ? And even if these properties/subgroups were not preserved by a certain isomorphism, could it be possible to fix this, that is, find another isomorphism between the tables of marks that does preserve them? In this paper we answer these and a few more questions concerning groups with isomorphic tables of marks.

In Section 2 we give the basic definitions and notation we shall use throughout this paper. In Section 3 we list the easier facts that can be deduced about a subgroup from the table of marks (such as being cyclic, maximal, the Frattini subgroup, the commutator subgroup, to name a few), and which must therefore be preserved by isomorphisms between tables of marks. In Section 4 we prove that elementary abelian subgroups can be identified from the table of marks (and must also be preserved by isomorphisms), including maximal elementary abelian subgroups. In Section 5 we give the smallest example (order-wise) of two non-isomorphic groups with isomorphic tables of marks, and conclude from this example that one cannot determine either abelian subgroups or the center of the group from the table of marks; we also observe that this problem is impossible to fix, in other words, there is neither an isomorphism between the tables of marks of these groups that preserves abelian subgroups, nor an isomorphism that makes the centres of these groups correspond. The same groups show that one cannot determine the normalizer of a subgroup from the table of marks, and that normalizers are not preserved by isomorphisms of tables of marks. In the last Section, we rely heavily on the data provided by GAP [1].

2 Tables of marks.

Definition 2.1. Let G, Q be finite groups. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G . We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$.

Given a subgroup H of G , we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#((Q/K')^{H'}) = \#((G/K)^H)$ for all subgroups H, K of G .

Note that $\#((G/K)^H) = \frac{|N_G(K)|}{|K|}\alpha(H, K) = \frac{|N_G(H)|}{|K|}\beta(H, K)$, where we define $\alpha(H, K)$ as the number of subgroups of G which are G -conjugate to K and contain H , and $\beta(H, K)$ equals the number of subgroups of K which are G -conjugate to H . If ψ is an isomorphism between the tables of marks of G and Q and we denote $\psi(H)$ by H' , we have that $|H'| = |H|$, $|N_Q(H')| = |N_G(H)|$, and $\alpha(H, K) = \alpha(H', K')$ for all H, K in $\mathfrak{C}(G)$; in fact, the previous conditions are an equivalent definition of an isomorphism between the tables of marks of G and Q . The matrix whose H, K -entry is $\#((G/K)^H)$ is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$). Some authors define the table of marks of G as the transpose of the previous matrix (for instance, that is how GAP defines it). Note that this matrix is defined up to an ordering of the elements of $\mathfrak{C}(G)$, so that the groups G and Q have isomorphic tables of marks if and only if it is possible to rearrange the elements of $\mathfrak{C}(G)$ and/or $\mathfrak{C}(Q)$ so that G and Q have identical tables of marks.

The **Burnside ring** of G , denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G . It is easy to see that if G and Q have isomorphic tables of marks, then they have isomorphic Burnside rings; the converse is an open problem (see [3]).

3 Preserved attributes.

An isomorphism between the tables of marks of two groups preserves many properties of the parent group and its subgroups. Here we list a few of these properties.

Theorem 3.1. *Let G, Q be finite groups with isomorphic tables of marks. Let K, H denote subgroups of G , and let K', H' denote representatives in their respective conjugacy classes of subgroups under the isomorphism between their tables of marks. Then we have that:*

1. $G' = Q, (1_G)' = 1_Q, |G| = |G'|, |H| = |H'|, \alpha(H, K) = \alpha(H', K'), \beta(H, K) = \beta(H', K'), |N_G(H)| = |N_Q(H')|.$

2. The subgroup H is normal in G if and only if H' is normal in Q . In this case, G/H and Q/H' have isomorphic tables of marks.
3. If $K \leq H$ and at least one of these two subgroups is normal in G , then $K' \leq H'$ for any choice of K' and H' .
4. If K and H are normal subgroups of G , then $(K \cap H)' = K' \cap H'$ and $(KH)' = K'H'$. In particular, two normal subgroups with trivial intersection correspond to two normal subgroups with trivial intersection. Furthermore, if $G = K \times H$, then $Q = K' \times H'$, K and K' have isomorphic tables of marks, and H and H' have isomorphic tables of marks.
5. The subgroup H is maximal in G if and only if H' is maximal in Q .
6. If G is a p -group, then $\text{socle}(Z(G))' = \text{socle}(Z(Q))$.
7. The Frattini subgroups correspond, that is, $\Phi(G)' = \Phi(Q)$.
8. The group G is nilpotent if and only if Q is nilpotent. However, there are non-isomorphic p -groups with isomorphic tables of marks.
9. For any divisor d of the order of H , the number of subgroups of H of order d is preserved; in particular, the total number of subgroups of H is preserved.
10. The subgroup H is cyclic if and only if H' is cyclic.
11. If H is isomorphic to the quaternion group of order 8, then H' is isomorphic to H .
12. If G is abelian then $G \cong Q$.
13. The commutator subgroups correspond, that is, $[G, G]' = [Q, Q]$. Moreover, the abelianized groups are isomorphic, that is, $G/[G, G] \cong Q/[Q, Q]$.
14. If G is isomorphic to S_n for some $n \geq 5$, then Q is isomorphic to G .
15. The subgroup H is elementary abelian if and only if H' is elementary abelian.

Proof. 1. This was observed before.

2. This follows from 1.
3. The normal subgroup corresponds to a unique subgroup; the rest follows from 1.
4. The intersection of two normal subgroups is the largest normal subgroup contained in both subgroups, KH is the smallest normal subgroups containing both K and H ; the rest is clear.
5. Assume H is a maximal subgroup of G . Let M' be a subgroup of Q between H' and Q , and let M be a corresponding subgroup in G . Since $0 \neq \alpha(H', M') = \alpha(H, M)$, a conjugate of M contains H . But H is maximal, so this conjugate is either H (so $M' = H'$) or G (so $M' = Q$).
6. Note that the socle of the centre of a p -group is characterized as the smallest normal subgroup of G such that has a nontrivial intersection with each nontrivial normal subgroup of G . This property is preserved under the correspondence.
7. Let $X = \Phi(G)$. By symmetry, it suffices to show that $X' \leq \Phi(Q)$. Note that X is a normal subgroup of G contained in all maximal subgroups. Since maximal subgroups correspond, then X' is a normal subgroup of Q contained in all maximal subgroups, so $X' \leq \Phi(Q)$.
8. Every Sylow p -subgroup of G is normal, and this property is preserved.
9. The number of subgroups of H of order d equals $\sum \beta(K, H)$ for all $K \in \mathfrak{C}(G)$ of order d .
10. A subgroup H is cyclic if and only if for each divisor d of $|H|$, H has exactly one subgroup of order d .
11. The quaternion group is the only group of order eight with three subgroups of order four.
12. The group G is a direct product of cyclic subgroups.
13. The commutator subgroup is the smallest normal subgroup of G with an abelian quotient. This property is preserved under the correspondence.

14. S_n with n greater than or equal to 5 is characterized by the following three properties: (1) It has order $n!$; (2) It only has one proper normal subgroup, whose order is $n!/2$; (3) It has a subgroup of index n (the action on the n cosets gives an isomorphism to S_n). These properties are preserved by an isomorphism between the tables of marks.
15. This will be done in the next Section. □

4 Elementary abelian subgroups.

In this Section we prove that an isomorphism between tables of marks sends elementary abelian subgroups to elementary abelian subgroups.

Lemma 4.1. *Let G be a finite group, p a prime number, a, b two elements in G of order p , A the subgroup generated by a and b . If A has order p^2 , then A is an elementary abelian group and $\#\{(c, d) \in G \times G \mid \langle c, d \rangle = A\} = \#\{(e, f) \in C_p^2 \times C_p^2 \mid \langle e, f \rangle = p^2\}$, where C_p^2 is the elementary abelian group of order p^2 .*

Proof. The group A is abelian because its order is p^2 . Since A is generated by two elements of order p that commute, A cannot be cyclic, so it has to be elementary abelian. Note that $\#\{(c, d) \in G \times G \mid \langle c, d \rangle = A\} = \#\{(x, y) \in A \times A \mid \langle x, y \rangle = A\} = \#\{(z, w) \in C_p^2 \times C_p^2 \mid \langle z, w \rangle = C_p^2\}$. □

Proposition 4.2. *Let p be a prime number, P, P' groups such that $|P| = |P'| = p^n$, with $n > 1$. Assume that P is elementary abelian, that all non-trivial elements of P' have order p , and that P' is not abelian. Let μ (respectively, μ') be the number of subgroups of P (respectively, P') of order p^2 . Then $\mu' < \mu$.*

Proof. Let $\Lambda = \{(a, b) \in P \times P \mid \langle a, b \rangle = p^2\}$, $\Lambda' = \{(a', b') \in P' \times P' \mid \langle a', b' \rangle = p^2\}$. Define equivalent relations $\tilde{\sim}$ in Λ and Λ' respectively by: $(a, b) \tilde{\sim} (c, d)$ iff $\langle a, b \rangle = \langle c, d \rangle$, and $(a', b') \tilde{\sim} (c', d')$ iff $\langle a', b' \rangle = \langle c', d' \rangle$. Note that by Lemma 4.1, each equivalence class in Λ has the same cardinality as each equivalence class in Λ' . Let τ denote this cardinality. Then we have that $\mu = \#\Lambda/\tilde{\sim} = \#\Lambda/\tau$, and $\mu' = \#\Lambda'/\tilde{\sim} = \#\Lambda'/\tau$. We must show that $\#\Lambda' < \#\Lambda$. Let $\Gamma = \{(a', b') \in P' \times P' \mid a' \neq 0, b' \notin \langle a' \rangle\}$. We shall prove that Λ' is a proper subset of Γ' and that Γ' and Λ have the same cardinality.

Let $(a', b') \in \Lambda'$. Since $|\langle a', b' \rangle| = p^2$, then $a' \neq 0$ (because $|\langle b' \rangle| = p$), and $b' \notin \langle a' \rangle$ (because $|\langle a' \rangle| = p$), that is, $(a', b') \in \Gamma'$. On the other hand, since P' is not abelian, there exist a', b' in P' which do not commute, so in particular we have that $|\langle a', b' \rangle| > p^2$, $a' \neq 0$, and $b' \notin \langle a' \rangle$, that is, (a', b') is in Γ' but not in Λ' .

Finally, note that both the elements of Γ' and Λ can be counted as follows: there are $p^n - 1$ ways to choose the first element, and $p^n - p$ ways to choose the second element, so each set has $(p^n - 1)(p^n - p)$ elements. \square

Corollary 4.3. *Let G and Q be finite groups with isomorphic tables of marks. If H is an elementary abelian subgroup of G , then H' is an elementary abelian subgroup of Q .*

Proof. We have that $|H'| = |H|$ is a power of a prime p . Since H has no cyclic subgroups of order p^2 , neither does H' , so H' is either elementary abelian or not abelian. If H' were not abelian, then H' would have strictly fewer subgroups of order p^2 than H . \square

5 Counterexamples.

We wrote software in GAP to go through the library of SmallGroups searching for the first instance of non-isomorphic groups with isomorphic tables of marks. The first known example before this computation was Thévenaz's pair of groups of order $5 * 11^2 = 625$ (see [5]). The smallest example has order 96.

Definition 5.1. Let M denote the direct product $S_3 \times C_8$ (that is, the symmetric group of degree 3, times the cyclic group of order 8). We shall give two non-isomorphic semidirect products of M with the cyclic group of order 2. Denote the elements of M as (σ, x^n) , where σ is a permutation in S_3 and x is the generator of C_8 . The group M is generated by the elements $((1, 2, 3), 1), ((1, 2), 1), ((1), x)$. Consider the following two automorphisms of order 2 of M , α and β , given by: $\alpha((1, 2, 3), 1) = \beta((1, 2, 3), 1) = ((1, 2, 3), 1), \alpha((1, 2), 1) = \beta((1, 2), 1) = ((1, 2), x^4), \alpha((1), x) = ((1), x)$ and $\beta((1), x) = ((1), x^5)$. Let G be the semidirect product of M with C_2 using α , and let Q be the semidirect product of M with C_2 using β . Both G and Q are groups of order 96, but they are not isomorphic; in fact, in GAP G is `SmallGroup(96, 108)`, and Q is `SmallGroup(96, 114)`.

Consider the following file written in GAP:

```
G := SmallGroup(96,108);
Q := SmallGroup(96,114);
#-----
testlattice := function(g,h)
# Explore the lattices of subgroups of the groups g and h.
  local lat, conj, ans, subg, lath, conjh, n, subh;

  lat := LatticeSubgroups(g);
  conj := ConjugacyClassesSubgroups(lat);
  lath := LatticeSubgroups(h);
  conjh := ConjugacyClassesSubgroups(lath);

  for n in [1..Size(conj)] do
    subg := ClassElementLattice(conj[n],1);
    subh := ClassElementLattice(conjh[n],1);

    if not(IsAbelian(subg)=IsAbelian(subh)) then
      Print("Subgroup number ",n,", Order ",Order(subg),
        " ",IsAbelian(subg)," ",IsAbelian(subh),"\\n");
    fi;
  od;
end;
#-----
subn := function(g,n)
# Return a representative of the n-th conjugcy class of subgroups
# of g.
  return ClassElementLattice(ConjugacyClassesSubgroups
    (LatticeSubgroups(g))[n],1);
end;
```

After loading this file, we checked that the tables of marks of G and Q are identical:

```
gap> Read("testiso");
gap> MatTom(TableOfMarks(G))=MatTom(TableOfMarks(Q));
true
```


This means that there is an isomorphism between the tables of marks of G and Q . Moreover, with the default tables of marks assigned by GAP, this isomorphism maps the n -th conjugacy class of subgroups of G to the n -th conjugacy class of subgroups of Q . We wonder whether the centres of G and Q correspond under this isomorphism.

```
gap> Size(Centre(G));
8
gap> Size(Centre(Q));
4
```

This proves that the centres of G and Q cannot correspond under this or any other isomorphism between their tables of marks (since such isomorphisms must preserve the order of the subgroups). Next we wonder whether abelian subgroups of G must necessarily correspond with abelian subgroups of Q .

The function `testlattice(G,Q)` runs through all the conjugacy classes of subgroups of G and Q (which are the same length), and it tests whether the corresponding subgroups are both abelian or both non-abelian. When it finds a pair of corresponding subgroups that do not match, it prints them on the screen, displaying their order and whether they are abelian or not.

```
gap> testlattice(G,Q);
Subgroup number 36, Order 16 true false
Subgroup number 37, Order 16 false true
Subgroup number 38, Order 16 false true
Subgroup number 40, Order 16 true false
Subgroup number 58, Order 48 true false
gap>
```

These are the only corresponding subgroups which are neither both abelian nor both non-abelian. Notice that there is exactly one abelian subgroup of G of order 48 which does not correspond to an abelian subgroup of Q (in fact, according to GAP, Q has no abelian subgroups of order 48). This means that G has exactly one more abelian subgroup of order 48 than Q , so there is no isomorphism between the tables of marks of G and Q that preserves abelian subgroups.

Finally, we show that the table of marks cannot provide enough information to determine the normalizer of a subgroup. Consider the function

`subn(g,n)`, which returns a representative of the n -th conjugacy class of subgroups of the group g .

```
gap> Normalizer(G,subn(G,2))=subn(G,58);
true
gap> Normalizer(Q,subn(Q,2))=subn(Q,58);
false
```

In both cases we had a subgroup in the second conjugacy class of subgroups; in G , its normalizer was the (only normal) subgroup in the 58-th conjugacy class, but in Q , the corresponding subgroup is not the normalizer.

We can summarize all this in the following result.

Theorem 5.2. *Let G and Q be finite groups with isomorphic tables of marks, and let $H \mapsto H'$ denote an isomorphism between their tables of marks. We have that*

1. H and H' may not be isomorphic.
2. Even if H is abelian, H' need not be abelian.
3. H and H' may have different tables of marks.
4. Even if $K \times L = H$, it may not be possible to find K' , L' and H' such that $K' \times L' = H'$.
5. Even if K is normal in H , it may not be possible to choose K' and H' such that K' is normal in H' .
6. Given H , the table of marks does not determine which subgroup of G is the normalizer of H in G .

Proof. Let G be `SmallGroup(96,108)` and Q be `SmallGroup(96,114)`.

1. This was known since Thévenaz's example, but it is also a consequence of our counterexample.
2. This is immediate.
3. This follows from the previous item and the fact that the table of marks determines an abelian group up to isomorphism.

4. If this were true, since cyclic subgroups correspond, it would follow that abelian subgroups map to abelian subgroups.
5. The subgroup $\text{subn}(G, 2)$ is a counterexample.
6. The subgroup $\text{subn}(G, 2)$ is again a counterexample.

□

Alberto G. Raggi-Cárdenas
Instituto de Matemáticas, UNAM, A.P. 61-3
C.P. 58089, Morelia, Mich, MEXICO
E-mail: graggi@matmor.unam.mx

Luis M. Huerta-Aparicio, Ariel Molina-Rueda, Luis Valero-Elizondo
Facultad de Ciencias Físico-Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio “B”, Planta Baja, Ciudad Universitaria
C.P. 58060, Morelia, Mich, MEXICO
E-mail: luherta@fismat.umich.mx, ariel@zarzamora.com.mx,
valero@fismat.umich.mx

References

- [1] The GAP Group. *GAP – Groups, Algorithms and Programming, Version 4.4*, 2006. (<http://www.gap-system.org>).
- [2] W. Kimmerle, Florian Luca, and Alberto G. Raggi-Cardenas. Irreducible components and isomorphisms of the Burnside ring. Submitted.
- [3] Florian Luca and Alberto G. Raggi-Cárdenas. Composition factors from the table of marks. *Journal of Algebra*, 244:737–743, 2001.
- [4] Alberto G. Raggi-Cárdenas and Luis Valero-Elizondo. Groups with isomorphic Burnside rings. *Archiv der Mathematik*, 84(3):193–197, 2005.
- [5] Jacques Thévenaz. Isomorphic Burnside rings. *Communications in Algebra*, 16(9):1945–1947, 1988.